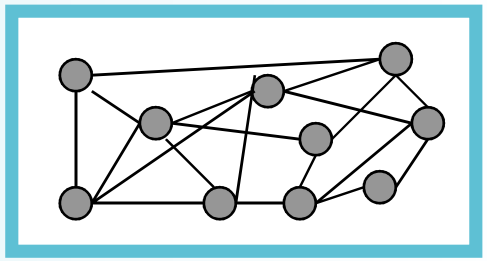
Discrete Mathematics

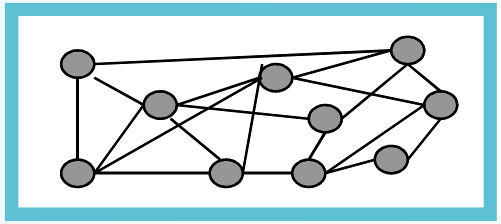
5. Graphs & Trees

What are Graphs?

- General meaning in everyday math:
  A plot or chart of numerical data using a coordinate system.
- Technical meaning in discrete mathematics:
  A particular class of discrete structures (to be defined) that is useful for representing relations and has a convenient webby-looking graphical representation.



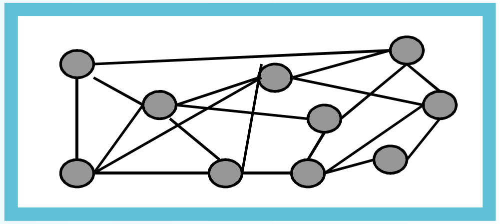
Simple Graphs



Visual Representation of a Simple Graph

- The graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices
- Correspond to symmetric binary relations R.

Simple Graphs



Visual Representation of a Simple Graph

- *Definition:*

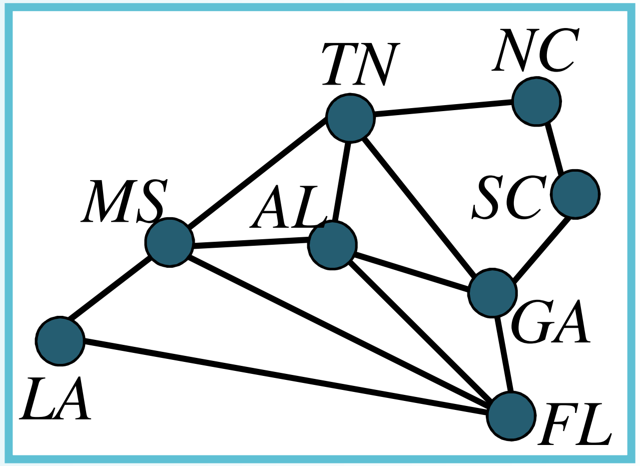
  A *simple graph* $G=(V,E)$ consists of:

  – a set V of *vertices* or *nodes* (V corresponds to the universe of the relation R), and

  – a set E of *edges* / *arcs* / *links*: unordered pairs of [distinct?] elements $u,v \in V$, such that uRv.

Example of a *Simple Graph*

- Let V be the set of states in the far-southeastern U.S.:
  V={FL, GA, AL, MS, LA, SC, TN, NC}
- Let E={{u,v}|u adjoins v}
  ={{FL,GA},{FL,AL},{FL,MS},
  {FL,LA},{GA,AL},{AL,MS},
  {MS,LA},{GA,SC},{GA,TN},
  {SC,NC},{NC,TN},{MS,TN},
  {MS,AL}}



Multigraphs

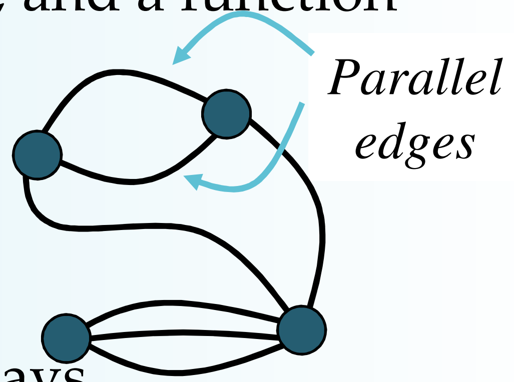
- Like simple graphs, but there may be *more than one* edge connecting two given nodes.
- *Definition*:

  A *multigraph* $G=(V, E, f)$ consists of a set V of vertices, a set E of edges (as primitive objects), and a function $f:E\rightarrow\{\{u,v\} \mid u,v\in V \wedge u\neq v\}$.

- *Example*:

  nodes are cities and

  edges are segments of major highways.



Pseudographs

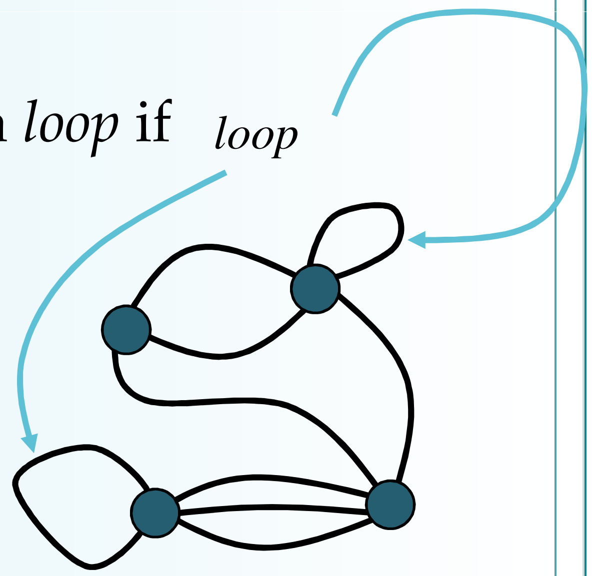
- Like a multigraph, but edges connecting a node to itself are allowed.

- *Definition*:

  A *pseudograph* $G=(V, E, f)$ where $f{:}E\rightarrow\{\{u,v\} \mid u,v\in V\}$. Edge $e\in E$ is a *loop* if $f(e)=\{u,u\}=\{u\}$.

- *Example:*

  nodes are campsites
  in a state park and edges are
  hiking trails through the woods.

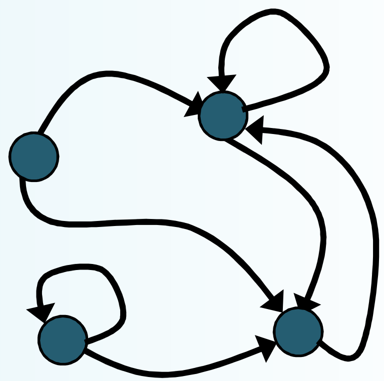


Directed Graphs

- Correspond to arbitrary binary relations R, which need not be symmetric.
- *Definition*:

  A *directed graph* (V, E) consists of a set of vertices V and a binary relation E on V.
- *Example:*

  V = people, $E=\{(x,y) \mid x \text{ loves } y\}$



Walk, loop, sling, and path

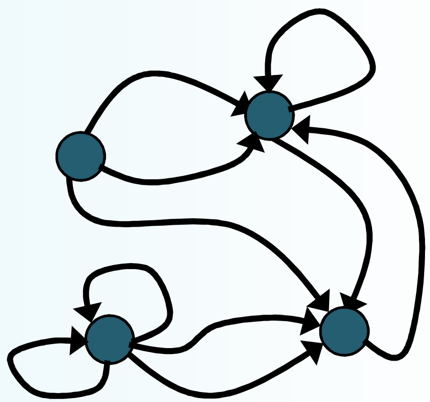
- **Definition:**
  - A ***walk*** is a sequence $x_0, x_1, \ldots, x_n$ of the vertices of a digraph such that $x_i x_{i+1}$, $0 \le i \le n-1$, is an edge.
  - The ***length of a walk*** is the number of edges in the walk.
  - If a walk holds $x_i \neq x_j$ $(i \neq j)$ $i,j=0, \ldots, n$, except x_0, x_n(i.e $x_0 = x_n$), the walk is called a ***cycle.***
  - A ***loop*** is a cycle of length one.
  - A ***sling*** is a cycle of length two.
  - A walk is a ***path*** if no edge is repeated more than once.

Directed Multigraphs

- Like directed graphs, but there may be more than one arc from a node to another.
- *Definition*:

  A *directed multigraph* $G=(V, E, f)$ consists of a set V of vertices, a set E of edges, and a function $f:E\rightarrow V\times V$.

- *Example*:
  - *The WWW is a directed multigraph.*
  - V=web pages, E=hyperlinks.



Types of Graphs: Summary

- Keep in mind this terminology is not fully standardized...

Term	Edge type	Multiple edges ok?	Self-loops ok?
Simple graph	Undir.	No	No
Multigraph	Undir.	Yes	No
Pseudograph	Undir.	Yes	Yes
Directed graph	Directed	No	Yes
Directed multigraph	Directed	Yes	Yes

Graph Terminology

- *Adjacent, connects, endpoints, degree, initial, terminal, in-degree, out-degree, complete, cycles, wheels, n-cubes, bipartite, subgraph, and union.*

Adjacency

Let G be an undirected graph with edge set E. Let $e \in E$ be (or map to) the pair $\{u,v\}$. Then we say:

- u, v are *adjacent* / *neighbors* / *connected*.
- Edge e is *incident with* vertices u and v.
- Edge e *connects* u and v.
- Vertices u and v are *endpoints* of edge e.

Degree of a Vertex

- Let G be an undirected graph, $v \in V$ a vertex.
- The *degree* of v, $\deg(v)$, is its number of incident edges. (Except that any self-loops are counted twice.)
- A vertex with degree 0 is *isolated*.
- A vertex of degree 1 is *pendant*.

Handshaking Theorem

- *Theorem*:
  - Let G be an undirected (simple, multi-, or pseudo-) graph with vertex set V and edge set E. Then
$$\sum_{v \in V} \deg(v) = 2|E|$$
- *Corollary*:
  - Any undirected graph has an even number of vertices of odd degree.

Directed Adjacency

- Let G be a directed (possibly multi-) graph, and let e be an edge of G that is (or maps to) (u,v). Then we say:
  - u is *adjacent to* v, v is *adjacent from* u
  - e *comes from* u, e *goes to* v.
  - e *connects* u *to* v, e *goes from* u *to* v
  - the *initial vertex* of e is u
  - the *terminal vertex* of e is v

Directed Degree

- *Definition*:

  Let G be a directed graph, v a vertex of G.

  - The *in-degree* of v, $\deg^{-}(v)$, is the number of edges going to v.
  - The *out-degree* of v, $\deg^{+}(v)$, is the number of edges coming from v.
  - The *degree* of v, $\deg(v) \equiv \deg^{-}(v) + \deg^{+}(v)$, is the sum of v's in-degree and out-degree.

Directed Handshaking Theorem

- *Theorem*:
  - Let G be a directed (possibly multi-) graph with vertex set V and edge set E. Then:

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = \frac{1}{2} \sum_{v \in V} \deg(v) = |E|$$

- Note that the degree of a node is unchanged by whether we consider its edges to be directed or undirected.

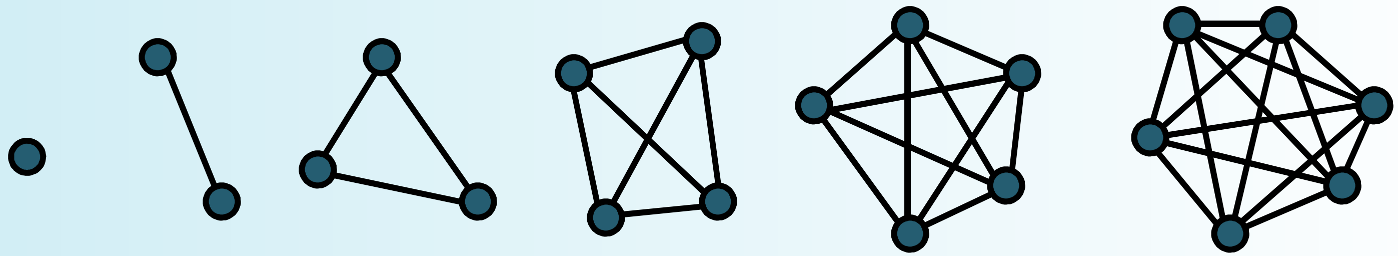
Special Graph Structures

Special cases of undirected graph structures:

- Complete Graphs K_n
- Cycles C_n
- Wheels W_n
- n-Cubes Q_n
- Bipartite Graphs
- Complete Bipartite Graphs $K_{m,n}$

Complete Graphs

- *Definition:*
  - For any $n \in \mathbf{N}$, a ***complete graph*** on n vertices, K_n, is a simple graph with n nodes in which every node is adjacent to every other node: $\forall u,v \in V$: $u \neq v \leftrightarrow \{u,v\} \in E$.

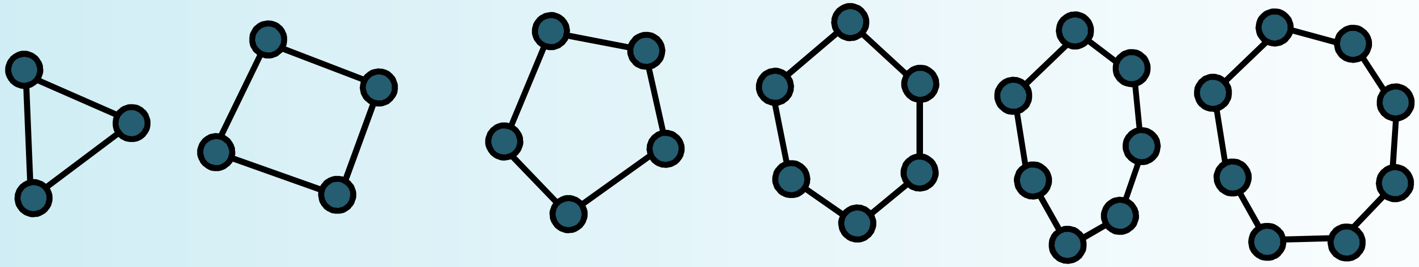


K_1 K_2 K_3 K_4 K_5 K_6

Note that K_n has $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ edges.

Cycles

- *Definition:*
  - For any $n \geq 3$, a ***cycle*** on n vertices, C_n, is a simple graph where $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\},\{v_n,v_1\}\}$.

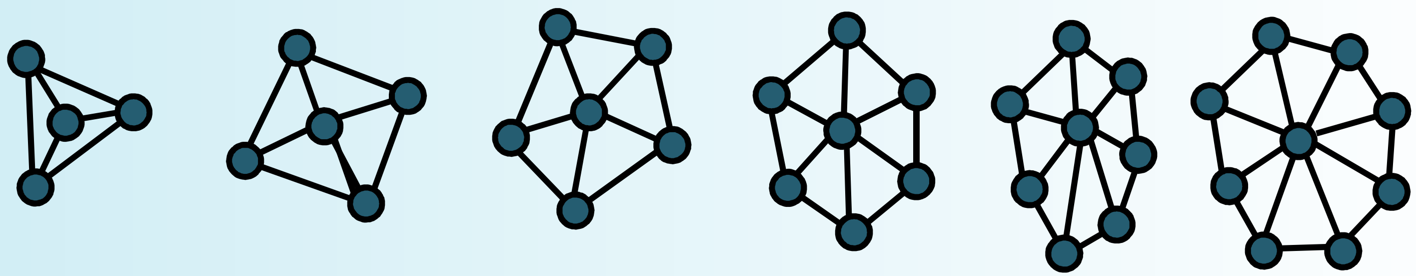


C_3 C_4 C_5 C_6 C_7 C_8

How many edges are there in C_n?

Wheels

- *Definition*:
  - For any $n \geq 3$, a ***wheel*** W_n, is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{\{v_{\text{hub}}, v_1\}, \{v_{\text{hub}}, v_2\}, \ldots, \{v_{\text{hub}}, v_n\}\}$.

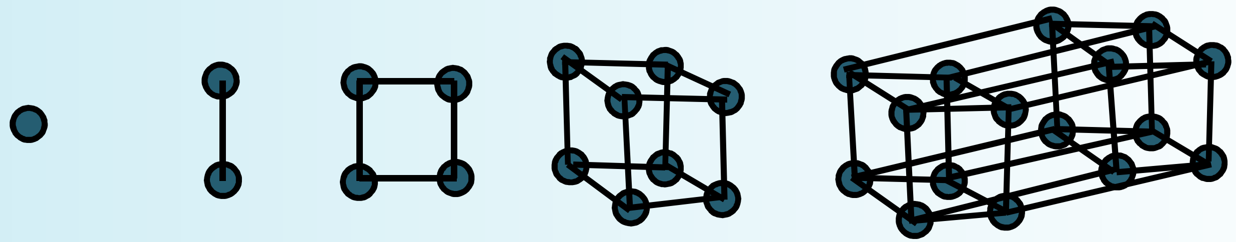


W_3 W_4 W_5 W_6 W_7 W_8

How many edges are there in W_n?

n-Cubes (hypercubes)

- *Definition:*
  - A graph that has vertices representing the 2^n bit strings of length n
  - For any $n \in \mathbf{N}$, the ***hypercube*** Q_n is a simple graph consisting of two copies of Q_{n-1} connected together at corresponding nodes. Q_0 has 1 node.



Q_0 Q_1 Q_2 Q_3 Q_4

Number of vertices: 2^n. *Number of edges:Exercise to try!*

n-Cubes (hypercubes)

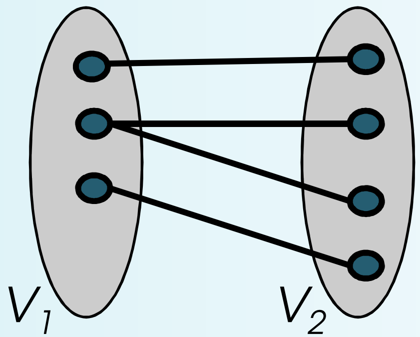
- *Definition*:

  For any $n \in \mathbf{N}$, the hypercube Q_n can be defined recursively as follows:

  - $Q_0=(\{v_0\},\varnothing)$ (one node and no edges)
  - For any $n \in \mathbf{N}$, if $Q_n=(V,E)$, where $V=\{v_1,\ldots,v_a\}$ and $E=\{e_1,\ldots,e_b\}$, then $Q_{n+1}=(V\cup\{v_1',\ldots,v_a'\}, E\cup\{e_1',\ldots,e_b'\}\cup\{\{v_1,v_1'\},\{v_2,v_2'\},\ldots, \{v_a,v_a'\}\})$ where $v_1',\ldots,v_a'$ are new vertices, and where if $e_i=\{v_j,v_k\}$ then $e_i'=\{v_j',v_k'\}$.

Bipartite Graphs

- *Definition*:
  - A simple graph G is called ***bipartite*** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2)



a bipartite

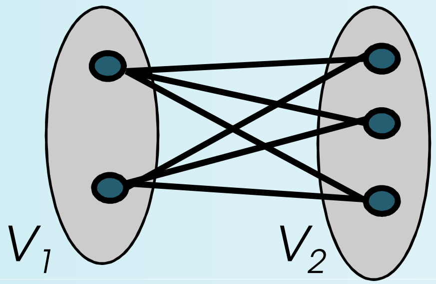
Complete Bipartite Graphs

- *Definition*:

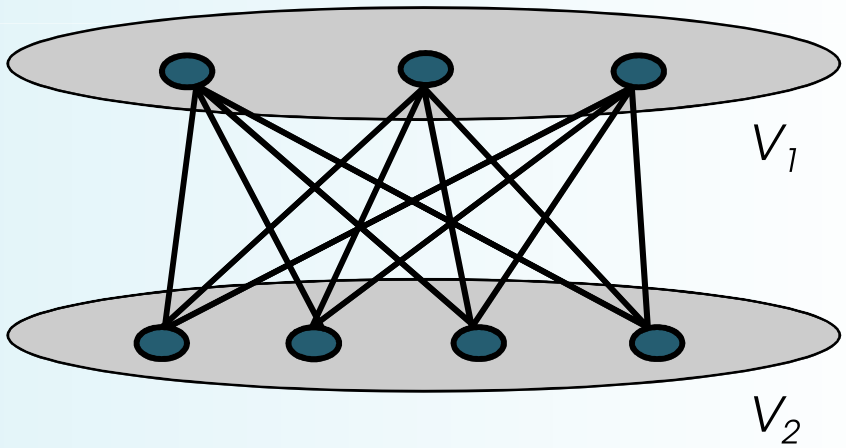
  Let m, n be positive integers. The ***complete bipartite graph*** $K_{m,n}$ is the graph whose vertices can be partitioned $V = V_1 \cup V_2$ such that

  1. $|V_1| = m$
  2. $|V_2| = n$
  3. For all $x \in V_1$ and for all $y \in V_2$, there is an edge between x and y
  4. No edge has both its endpoints in V_1 or both its endpoints in V_2

Complete Bipartite Graphs (cont.)



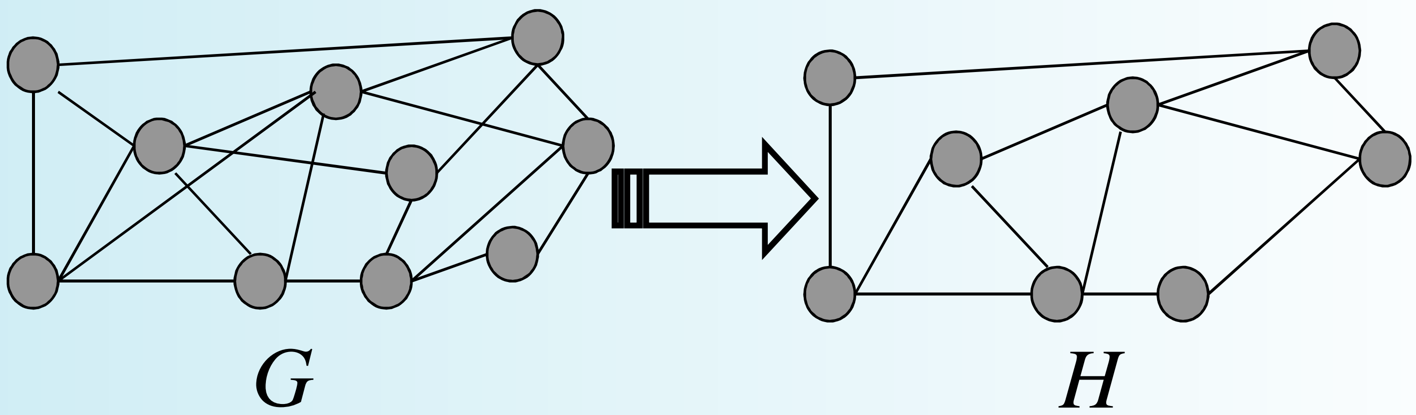
$K_{2,3}$



$K_{3,4}$

Subgraphs

- *Definition*:
  - A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$ where $W \subseteq V$ and $F \subseteq E$.



Graph Unions

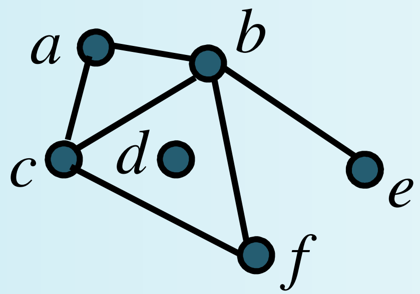
- *Definition*:
  - The *union* $G_1 \cup G_2$ of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2,E_2)$ is the simple graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Graph Representations & Isomorphism

- Graph representations:
  - Adjacency lists.
  - Adjacency matrices.
  - Incidence matrices.
- Graph isomorphism:
  - Two graphs are isomorphic iff they are identical except for their node names.

Adjacency Lists

- A table with 1 row per vertex, listing its adjacent vertices.
- A way to represent a graph w/ no multiple edges



Vertex	*Adjacent Vertices*
a	*b*, *c*
b	*a*, *c*, *e*, *f*
c	*a*, *b*, *f*
d	
e	*b*
f	*c*, *b*

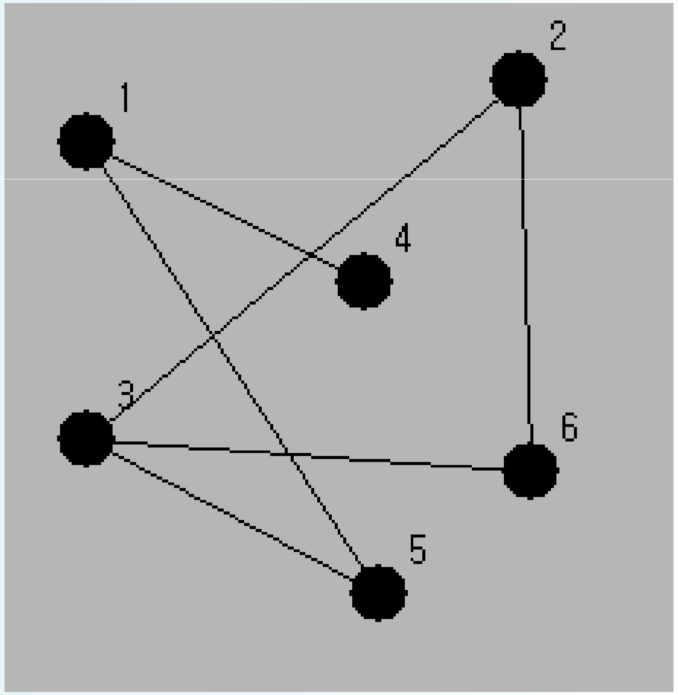
Directed Adjacency Lists

- 1 row per node, listing the terminal nodes of each edge incident from that node.

Adjacency Matrices

- Matrix **A**=[a_{ij}], where a_{ij} is 1 if {v_i, v_j} is an edge of *G*, 0 otherwise.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$



- Simple graph representation

Adjacency Matrices

- Matrix $\mathbf{M}=[m_{ij}]$, where
  - $m_{ij} = 1$ when edge e_j is incident with v_i,
  - $m_{ij} = 0$ otherwise.
- A way of represent graphs
  - can be used to represent multiple edges and loops

Graph Isomorphism

- *Definition*:

  Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* iff $\exists$ a bijection $f:V_1 \rightarrow V_2$ such that $\forall$ $a,b \in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.

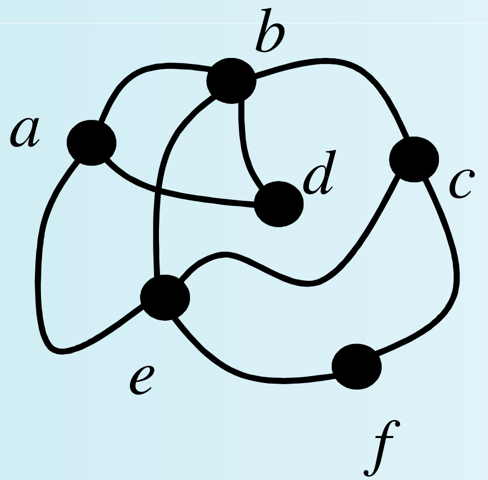
- f is the "renaming" function that makes the two graphs identical.
- Definition can easily be extended to other types of graphs.

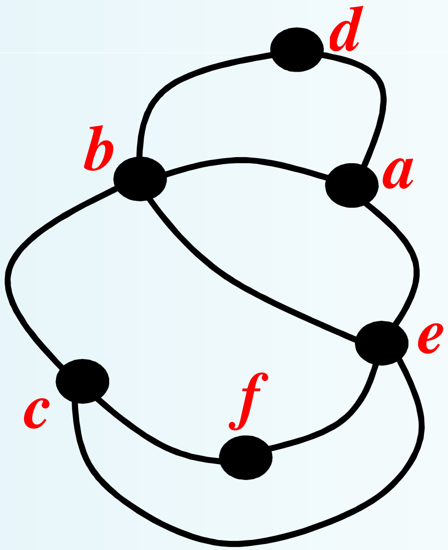
Graph Invariants under Isomorphism

- Graph Invariant
  - a property preserved by isomorphism of graphs
  - *Necessary* but not *sufficient* conditions for $G_1=(V_1, E_1)$ to be isomorphic to $G_2=(V_2, E_2)$:
    - $|V_1|=|V_2|$, $|E_1|=|E_2|$.
    - The number of vertices with degree n is the same in both graphs.
    - For every proper subgraph g of one graph, there is a proper subgraph of the other graph that is isomorphic to g.

Isomorphism Example

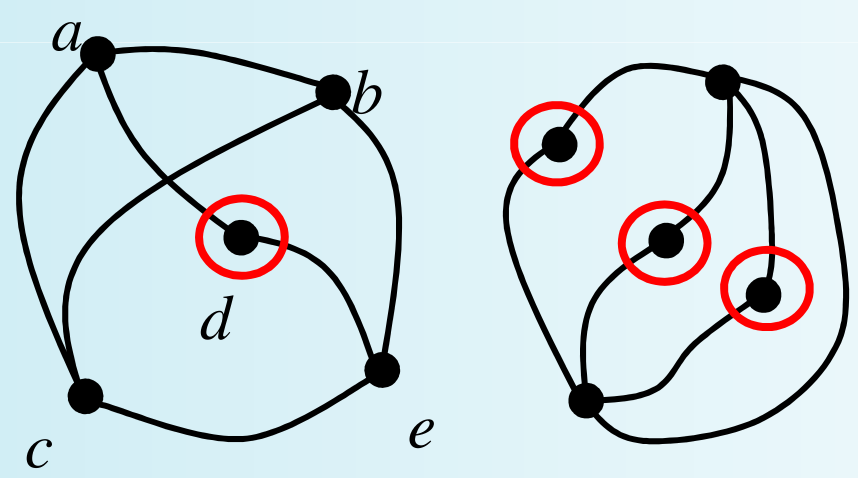
- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.





Are These Isomorphic?

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



** Same # of vertices*

** Same # of edges*

** Different # of verts of degree 2! (1 vs 3)*

Connectedness

- *Definition*:

  A *path of length n* from u to v in G is

  a sequence of n edges $e_1, e_2, \ldots, e_n$ such that e_1 is associated with $\{x_0, x_1\}$, e_2 is associated with $\{x_1, x_2\}$, and so on, with e_n associated with $\{x_{n-1}, x_n\}$, where $x_0=u$ and $x_n=v$;

- The path is *circuit* if it begins and ends at the same vertex, that is $u=v$, and has length greater than zero.
- A path or circuit is *simple* if it does not contain the same edge more than once.

Connectedness

- *Definition*:
  - An undirected graph is *connected* iff there is a path between every pair of distinct vertices in the graph.
- *Theorem*:
  - There is a *simple* path between any pair of vertices in a connected undirected graph.

Directed Connectedness

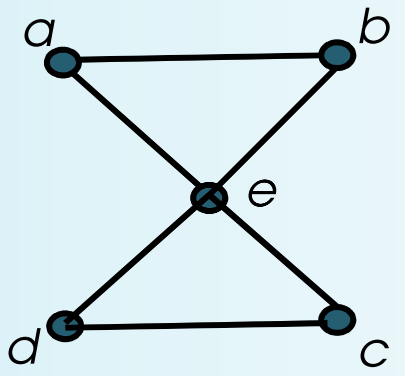
- *Definition*:
  - A directed graph is *strongly connected* iff there is a directed path from *a* to *b* for any two vertices *a* and *b*.
  - It is *weakly connected* iff the underlying *undirected* graph (*i.e.*, with edge directions removed) is connected.
- Note *strongly* implies *weakly* but not vice-versa.

Paths & Isomorphism

- Note that connectedness, and the existence of a circuit or simple circuit of length k are graph invariants with respect to isomorphism.

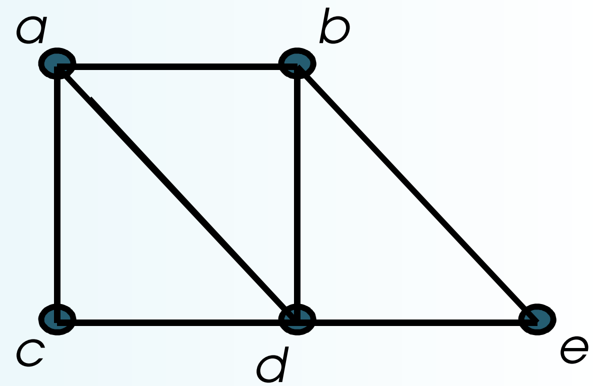
Euler & Hamilton Paths

- *Definition:*
  - An *Euler circuit* in a graph G is a simple circuit containing every edge of G.
  - An *Euler path* in G is a simple path containing every edge of G.
- *Examples*:



a, e, c, d, e, b, a

Euler circuit

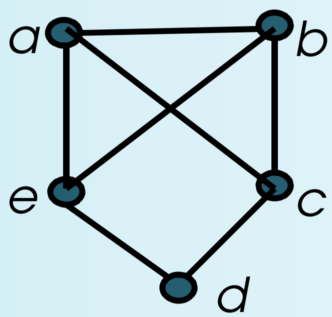


a, c, d, e, b, d, a, b

Euler path

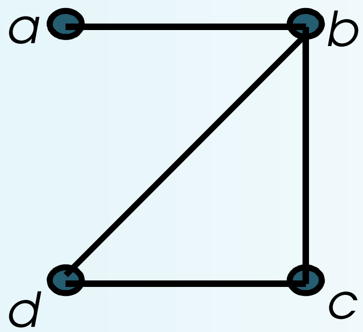
Euler & Hamilton Paths (cont.)

- *Definition*:
  - A *Hamilton circuit* is a simple circuit that traverses each vertex in G exactly once.
  - A *Hamilton path* is a simple path that traverses each vertex in G exactly once.
- *Examples*:



a, b, c, d, e, a

Hamilton circuit



a, b, c, d

Hamilton path

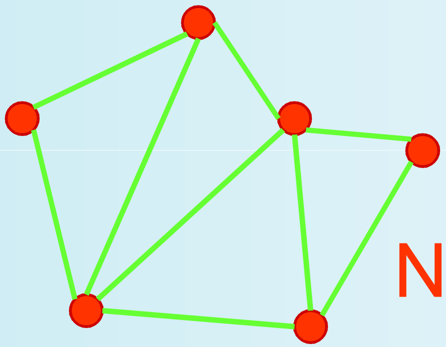
Trees

- *Definition*:
  - A **tree** is a connected undirected graph with no simple circuits.
- Since a tree cannot have a circuit, a tree cannot contain multiple edges or loops.
- Therefore, any tree must be a **simple graph**.

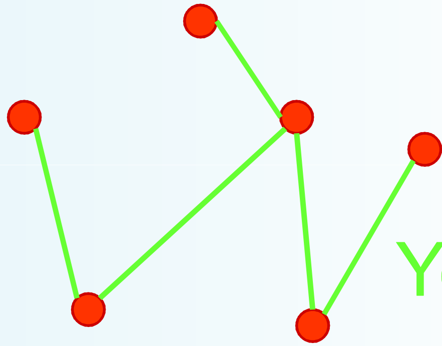
- *Theorem*:
  - An undirected graph is a tree if and only if there is a **unique simple path** between any two of its vertices.
- In general, we use trees to represent **hierarchical structures**.

Trees

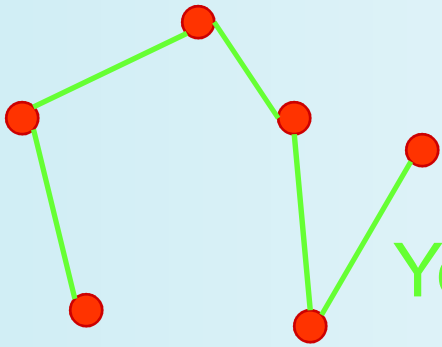
Example: Are the following graphs trees?



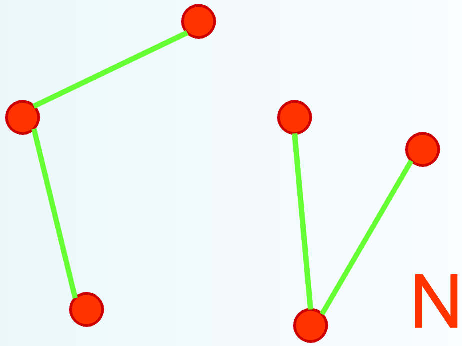
No.



Yes.



Yes.



No.

Root & Rooted tree

- We often designate a particular vertex of a tree as the **root**. Since there is a unique path from the root to each vertex of the graph, we direct each edge away from the root.

- *Definition*:

  A **rooted tree** is a tree in which one vertex has been designated as the root and every edge is directed away from the root

Tree Terminology

- *Definition*:
  - If v is a vertex in a rooted tree other than the root, the **parent** of v is the unique vertex u such that there is a directed edge from u to v.
  - When u is the parent of v, v is called the **child** of u.
  - Vertices with the same parent are called **siblings**.
  - The **ancestors** of a vertex other than the root are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root.

Tree Terminology (cont.)

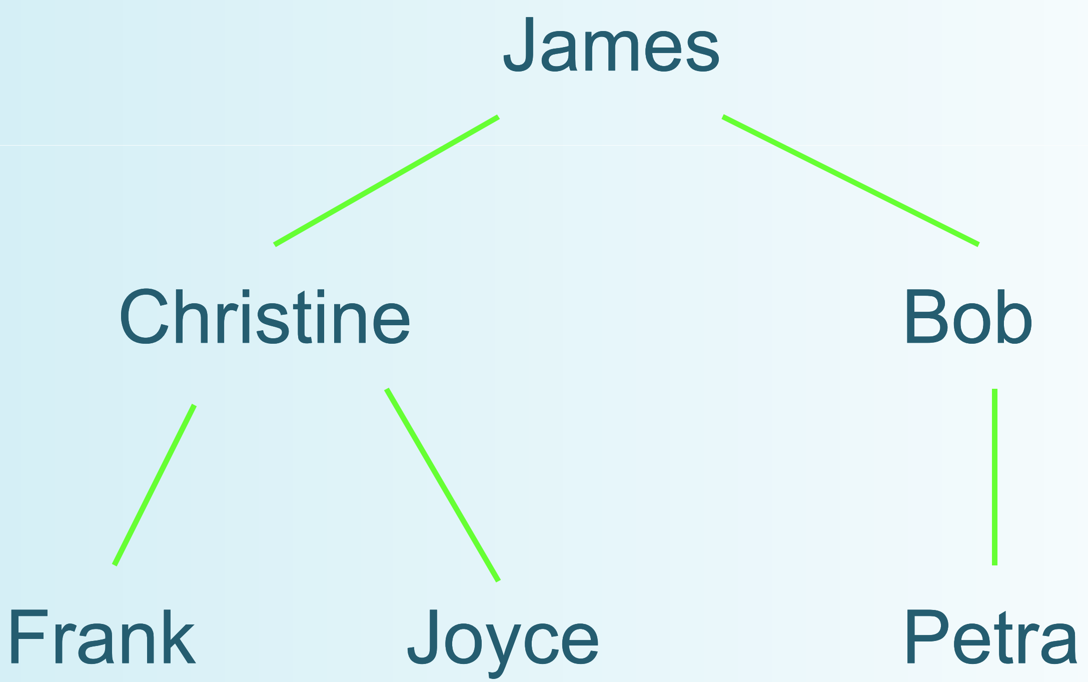
- *Definition*:
  - The **descendants** of a vertex v are those vertices that have v as an ancestor.
  - A vertex of a tree is called a **leaf** if it has no children.
  - Vertices that have children are called **internal vertices**.
  - If a is a vertex in a tree, then the **subtree** with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.

Tree Terminology

- *Definition*:
  - The **level** of a vertex v in a rooted tree is the length of the unique path from the root to this vertex.
  - The level of the root is defined to be zero.
  - The **height** of a rooted tree is the maximum of the levels of vertices.

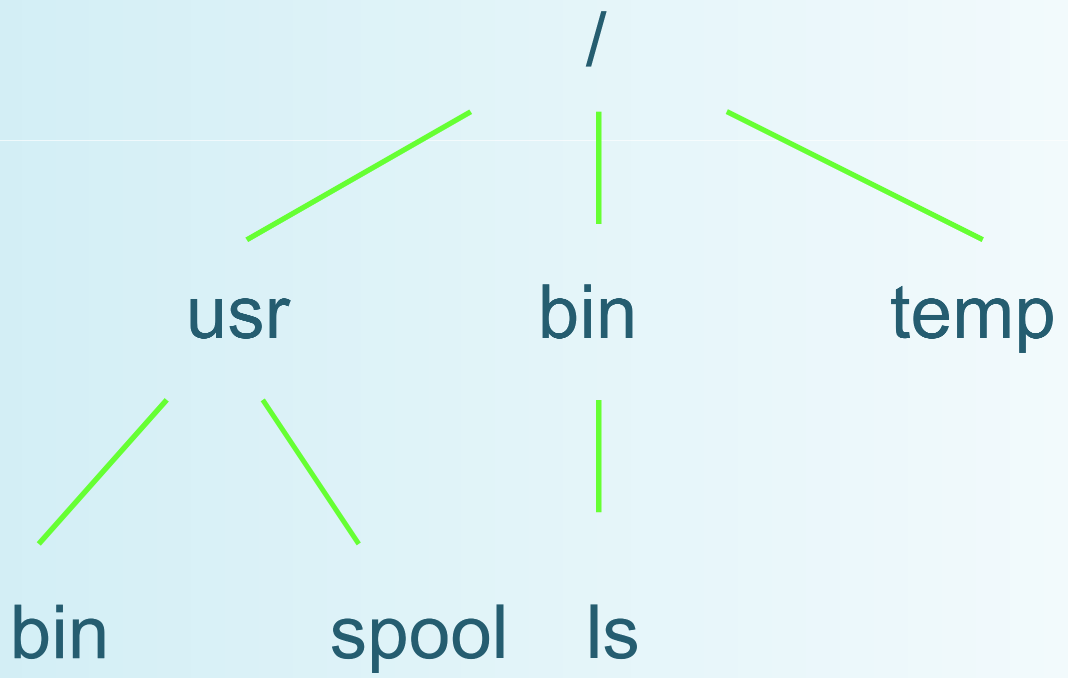
Trees

Example 1: Family tree



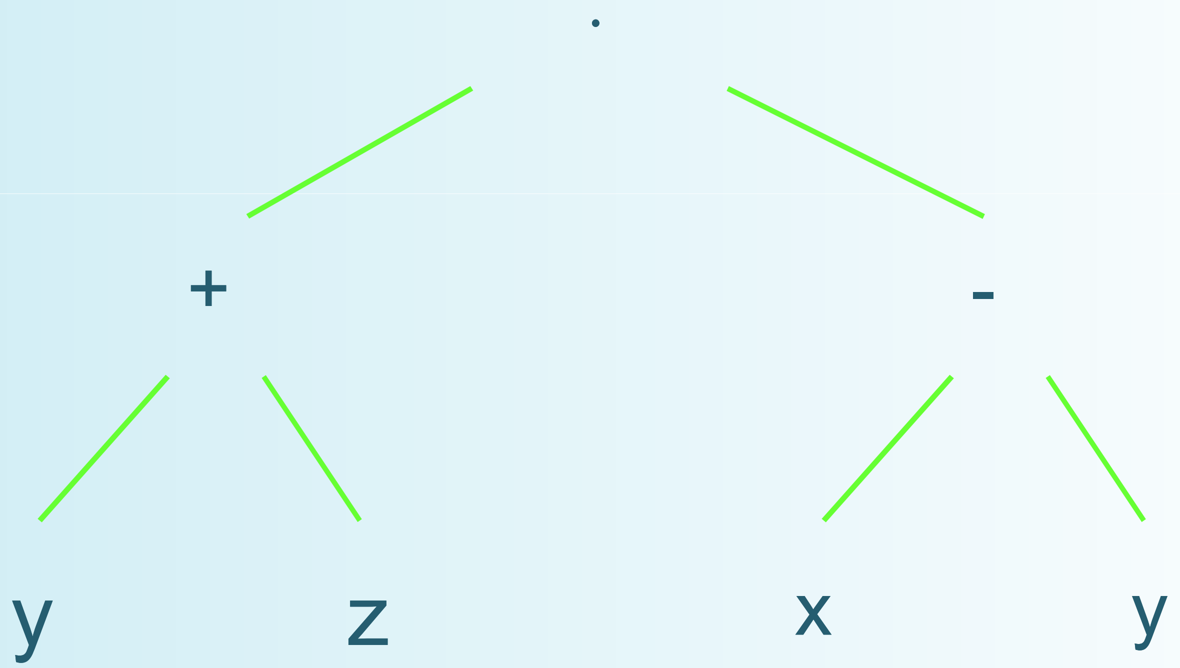
Trees (cont.)

Example 2: File system



Trees (cont.)

Example 3: Arithmetic expressions



- This tree represents the expression (y + z)·(x - y).

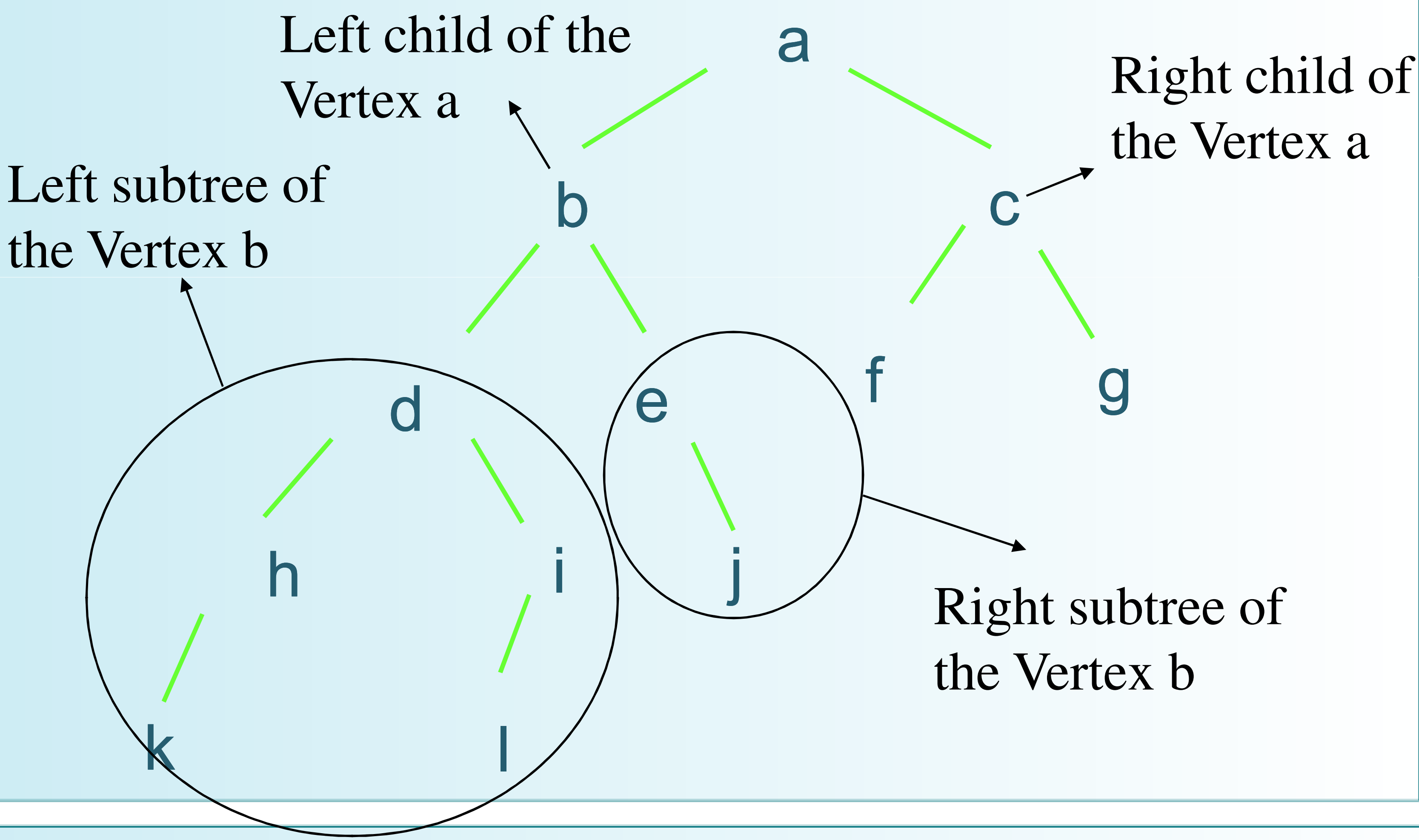
m-ary tree

- *Definition*:
  - A rooted tree is called an ***m*-ary tree** if every internal vertex has no more than m children.
  - The tree is called a **full *m*-ary tree** if every internal vertex has exactly m children.
  - An m-ary tree with $m = 2$ is called a **binary tree**.
- *Theorem*:
  - A tree with n vertices has $(n - 1)$ edges.
  - A full m-ary tree with i internal vertices contains $n = m \cdot i + 1$ vertices.

Ordered Rooted Tree

- *Definition*:
  - An **ordered rooted tree** is a rooted tree where the children of each internal vertex are ordered.
- For example, in an ordered binary tree (just called binary tree), if an internal vertex has two children,
  - the first child is called the left child and the second is called the right child.
  - the tree rooted at the left child is called the left subtree, and at the right child, the right subtree
- Ordered rooted trees can be defined recursively.

Ordered Rooted Tree (binary tree)



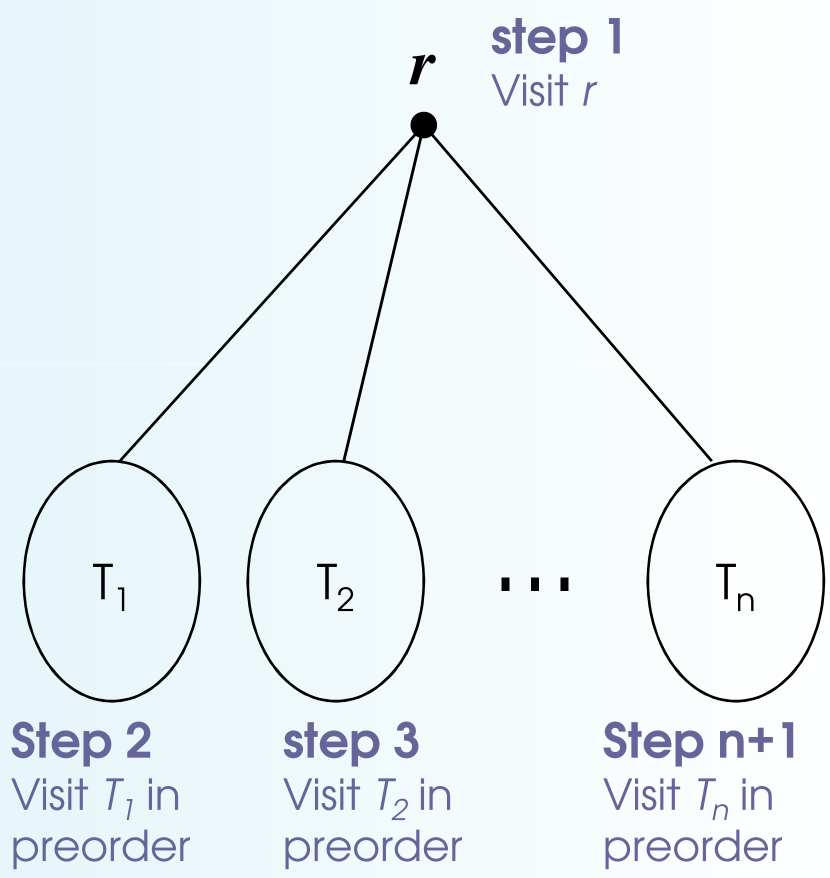
Tree Traversal

- Procedures for systematically visiting every vertex of an ordered rooted tree are called traversal algorithms.

Preorder traversal

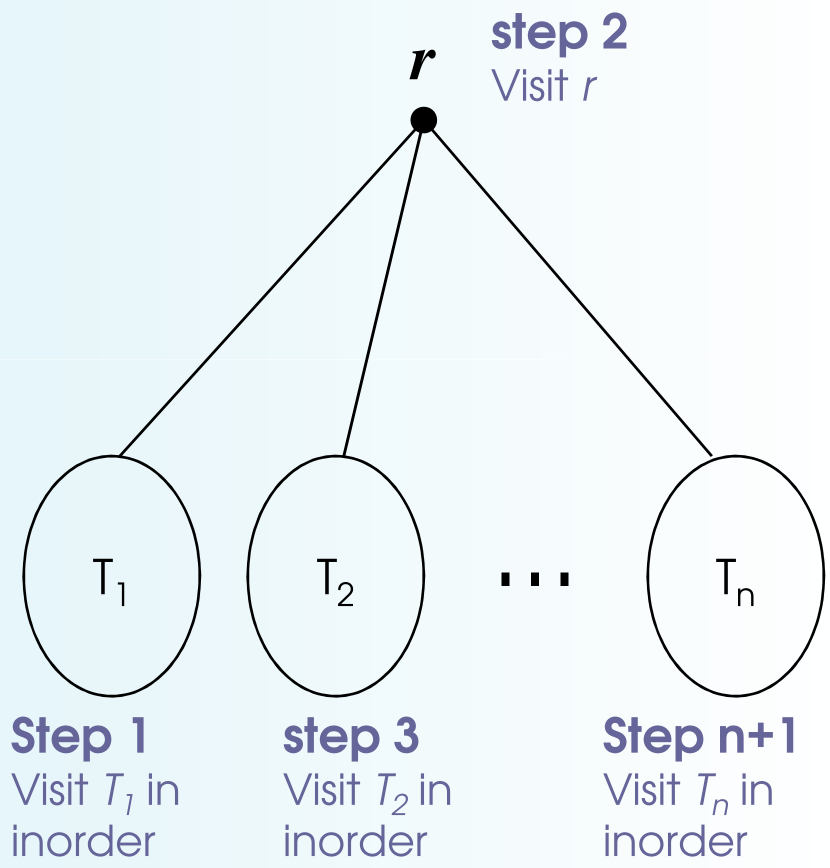
- *Definition*:

  Let T be an ordered rooted tree with root r. If T consists only of r, then r is the *preorder traversal* of T. Otherwise, suppose that T_1, T_2, …, T_n are the subtrees at r from left to right in T. The *preorder traversal* begins by visiting r. It continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.



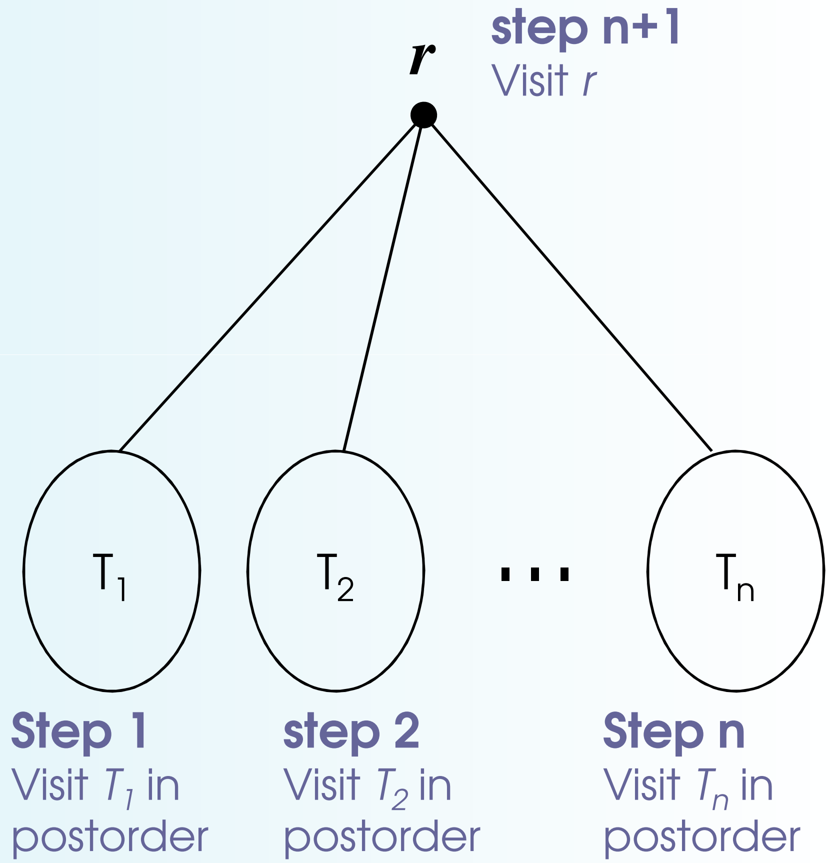
Inorder traversal

- *Definition*:
  Let T be an ordered rooted tree with root r.
  If T consists only of r, then r is the *inorder traversal* of T.
  Otherwise, suppose that T_1, T_2, …, T_n are the subtrees at r from left to right. The *inorder traversal* begins by traversing visiting T_1 in inorder, then visiting r. It continues by traversing T_2 in inorder, then T_3 in inorder, …, and finally T_n in inorder

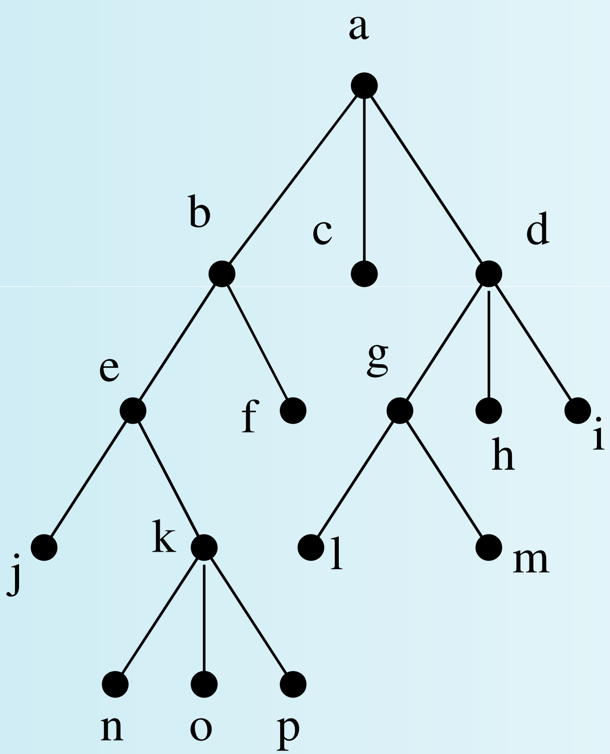


Postorder traversal

- *Definition*:
  Let T be an ordered rooted tree with root r.
  If T consists only of r, then r is the *postorder traversal* of T.
  Otherwise, suppose that T_1, T_2, …, T_n are the subtrees at r from left to right. The *postorder traversal* begins by traversing T_1 in postorder, then T_2 in postorder, …, then T_n in postorder, and ends by visiting r.



Traversal Example



- ***Preorder*** : a, b, e, j, k, n, o, p, f, c, d, g, l, m, h, I
- ***Inorder*** : j, e, n, k, o, p, b, f, a, c, l, g, m, d, h, i
- ***Postorder*** : j, n, o, p, k, e, f, b, c, l, m, g, h, i, d, a