Discrete Mathematics

3. Functions

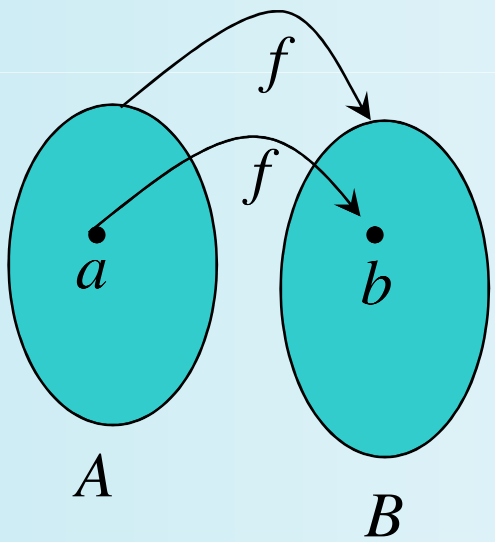
Function

- *Definition :*
  - Let A and B be two sets. A relation f from A to B is called a function iff for every x in A, there is a unique y in B such that $\langle x, y\rangle \in f$

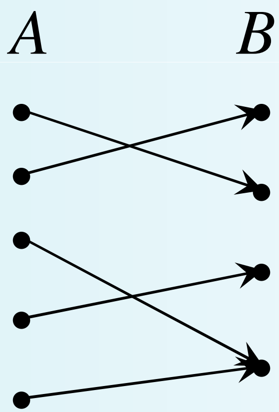
- A relation $f \subseteq A \times B$ may be written by $f : A \rightarrow B$ and $\langle x, y\rangle \in f$ by $f(x)=y$ if f is a function.

Graphical Representations

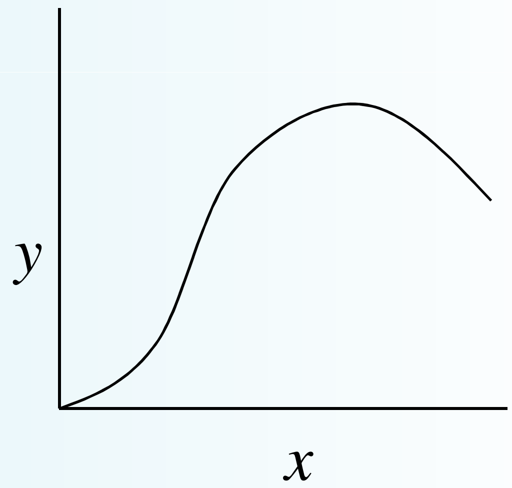
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Some Function Terminology

- *Definition:*

  If $f:A\rightarrow B$, and $f(a)=b$ (where $a\in A$ & $b\in B$), then:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R\subseteq B$ of f is $\{b \mid \langle a, b\rangle \in f$ for some $a\}$.

Images of Sets under Functions

- *Definition:*

  Given f:$A \rightarrow B$, and $S \subseteq A$, the *image* of S under f is defined to be the set of all images (under f) of the elements of S.

  $f(S) :\equiv \{ f(s) \mid s \in S\}$

- Note the range of f can be defined as simply the image (under f) of f's domain!

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the *particular* set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {A,B,C,D,E}."
- At this point, you know *f*'s codomain is: {A,B,C,D,E}, and its range is unknown!
- Suppose the grades turn out all As and Bs.
- Then the range of *f* is {A,B}, but its codomain is still {A,B,C,D,E}!

Restriction and extension

- *Definition:*
  - If $f: X \to Y$ and $A \subseteq X$, then $f \cap (A \times Y)$ is a function from A to Y called the ***restriction*** of f to A and is sometimes written as f/A.
  - If g is a restriction of f, then f is called the ***extension*** of g.
  - $(f/A): A \to Y, \forall a \in A, (f/A)(a)=f(a)$

Operators (general definition)

- *Definition:*
  - An n-ary *operator* O_n over the set S is any function from the set of ordered n-tuples of elements of S, to S itself.

$$O_n: S^n \to S$$

- *Example:*
  - If S={**T**,**F**}, ¬ can be seen as a unary operator, and ∧,∨ are binary operators on S.
  - ∪ and ∩ are binary operators on the set of all sets.

Constructing Function Operators

- If • ("dot") is any operator over B, then we can extend • to also denote an operator over functions $f{:}A\rightarrow B$.
- *Example:*
  - Given any binary operator $\bullet{:}B\times B\rightarrow B$, and functions $f, g{:}\ A\rightarrow B$, we define $(f \bullet g){:}A\rightarrow B$ to be the function defined by:
    $\forall a\in A,\ (f \bullet g)(a) = f(a)\bullet g(a)$.

Example of Function Operator

- $+,\times$ ("plus","times") are binary operators over $\mathbf{R}$. (Normal addition & multiplication.)
- Therefore, we can also add and multiply *functions* $f,g:\mathbf{R}\rightarrow\mathbf{R}$:
  - $(f+g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f+g)(x) = f(x) + g(x)$
  - $(f\times g):\mathbf{R}\rightarrow\mathbf{R}$, where $(f\times g)(x) = f(x)\times g(x)$

Function Composition

- *Definition:*
  - Let $g:A \rightarrow B$ and $f:B \rightarrow C$ be two functions. Then the function composition, $f \circ g$, from A to C is

    $f \circ g = \{ <x,y> \mid (\exists z)((<x,z> \in g) \wedge (<z,y> \in f)) \}$

- Note that $\circ$ (like Cartesian $\times$, but unlike $+, \wedge, \cup$) is non-commuting. (Generally, $f \circ g \neq g \circ f$)

Composition of Functions

- *Theorem:*
  - Let $g:A \to B$ and $f:B \to C$ be functions. Then the function composition $f \circ g$ is a function from A to C and $(f \circ g)(a) = f(g(a))$ for all $a \in A$
- *Theorem:*
  - Composition of function is associative: If f, g and h are functions, then $(f \circ g) \circ h = f \circ (g \circ h)$

Partial Function

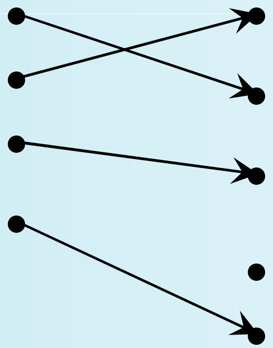
- *Definition:*
  - Let X and Y be sets. A ***partial function*** f with domain X and codomain Y is any function from X' to Y, where $X' \subseteq X$. For any $x \in X - X'$, the value of $f(x)$ is said to be ***undefined***.

One-to-One Functions

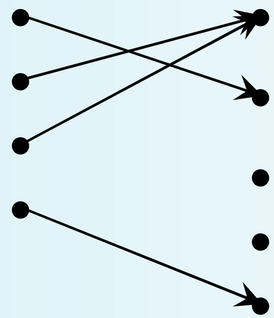
- *Definition :*
  - A function $f : A \to B$ is *one-to-one* (1-1), or *injective,* or *an injection,* iff every element of its range has *only* 1 pre-image
  - for every x_1 and x_2 in A, if $f(x_1) = f(x_2)$, then $x_1 = x_2$
  - for every x_1 and x_2 in A, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

One-to-One Illustration

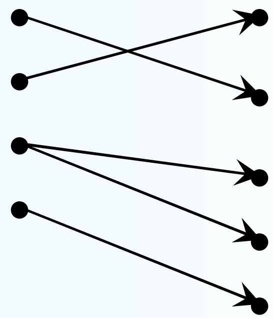
- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one



Not one-to-one



Not even a function!

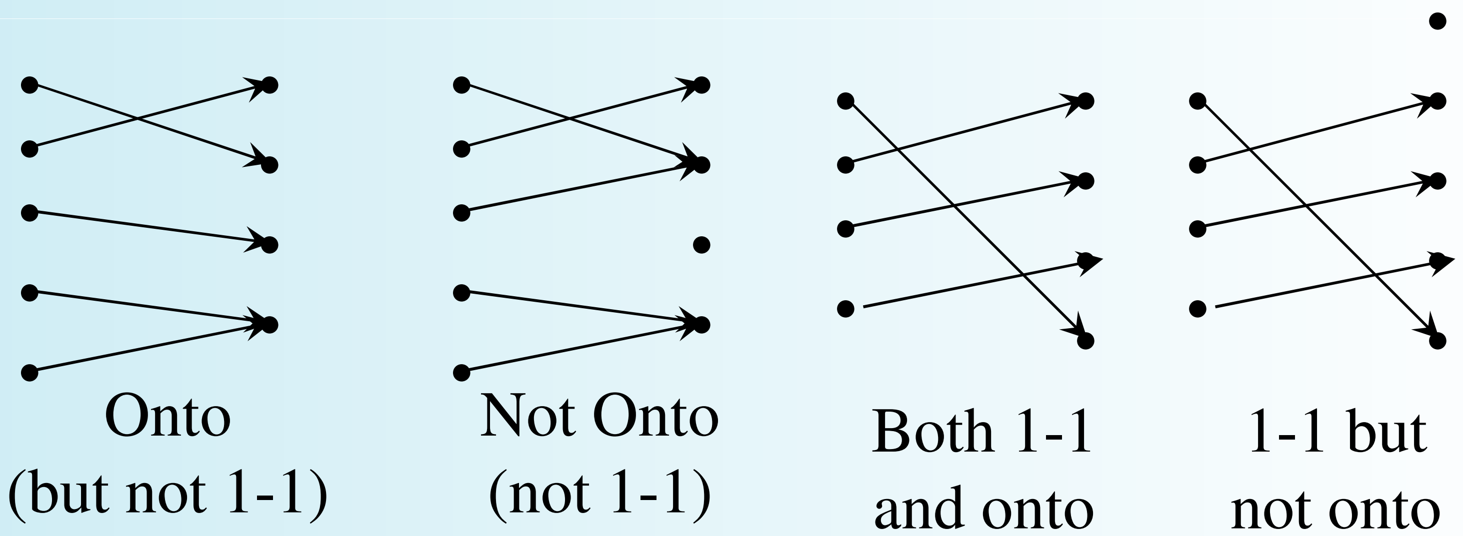
Onto (Surjective) Functions

- *Definition* **:**
  - A function $f : A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff for every b in B, there exists a in A such that $f(a)=b$.

- *Example:*
  - For domain & codomain **R**, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are or are not *onto* their codomains:



Bijections

- *Definition* **:**
  - A function $f : A \rightarrow B$ is *a one-to-one correspondence,* or *a bijection,* or ***bijective*** *reversible,* or *invertible,* iff it is both one-to-one and onto.

Surjective, Injective, Bijective

- *Theorem:*

  Let $f \circ g : A \to C$ be a composite function where g: $A \to$B and f: $B \to C$.

  (a) If f and g are surjective then $f \circ g$ is surjective.

  (b) If f and g are injective then $f \circ g$ is injective.

  (c) If f and g are bijective then $f \circ g$ is bijective.

  (d) If $f \circ g$ is surjective then f is surjective.

  (e) If $f \circ g$ is injective then g is injective.

  (f) If $f \circ g$ is bijective then f is surjective and g is injective.

Constant Function

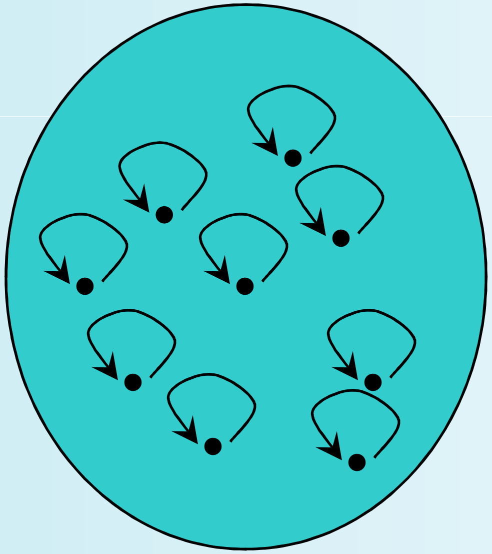
- *Definition:*
  - Let a function f: $X \rightarrow Y$ is a ***constant function*** if there exist some $y \in Y$ such that $f(x)=y$ for every x in X.

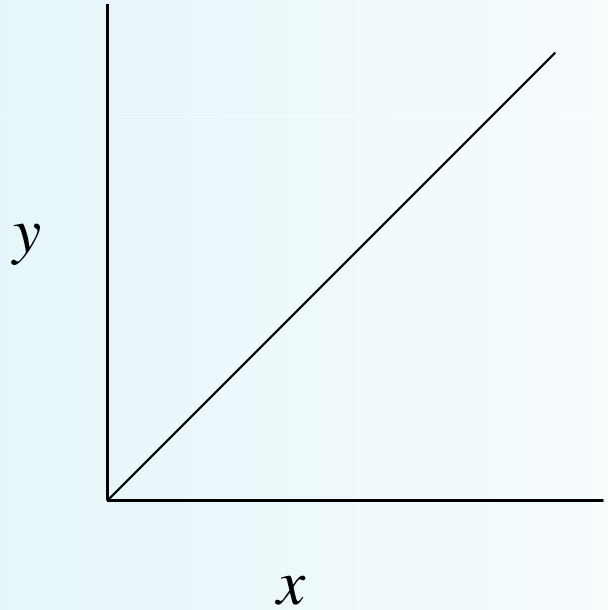
Identity Function

- *Definition:*
  - For any domain A, the *identity function* $I{:}A \to A$ (variously written, I_A, $\mathbf{1}$, $\mathbf{1}_A$) is the unique function such that $\forall a \in A$, $I(a)=a$.
- Some identity functions you've seen:
  - $+$ing 0, $\cdot$ing by 1, $\wedge$ing with **T**, $\vee$ing with **F**, $\cup$ing with $\emptyset$, $\cap$ing with U.
- Note that the identity function is both one-to-one and onto (bijective).
- Let f: $X \to Y$ then $f = f \circ I_X = I_Y \circ f$

Identity Function Illustrations

- The identity function:


Domain and range



Inverse Function

- *Definition:*
  - Let $f{:}X \rightarrow Y$ is bijection from X to Y. The ***inverse function*** of f, denoted by f^{-1} is the converse relation of f.
- *Theorem:*
  - Let f be a bijective function $f{:}X \rightarrow Y$. Then f^{-1} is a bijective function and $f^{-1}{:}\ Y \rightarrow X$
  - If f is bijective $(f^{-1})^{-1} = f$.

Inverse Function (cont.)

- *Definition:*
  - Let $h: A \rightarrow B$ and $g: B \rightarrow A$. If $g \circ h = I_{A}$, then g is a ***left inverse*** of h and h is a ***right inverse*** of g.
- *Theorem:*

  Let $f: A \rightarrow B$ with $A \neq \phi$, then

  (a) f has a left inverse if and only if f is injective

  (b) f has a right inverse if and only if f is surjective

  (c) f has a left and right inverse if and only if f is bijective.

  (d) If f is bijective, then the left and right inverses of f are equal.

A Couple of Key Functions

- In discrete math, we will frequently use the following functions over real numbers:
  - $\lfloor x \rfloor$ ("floor of x") is the largest integer $\leq x$.
  - $\lceil x \rceil$ ("ceiling of x") is the smallest integer $\geq x$.

Finite Set and Cardinality

- *Definition:*
  - A set A is ***finite*** if there is some natural number $n \in N$ such that there is a bijection from the set {1, 2, …, n} to the set A.
  - The integer n is called the ***cardinality*** of A, and we say "The set A has n elements," or " n is the ***cardinal number*** of A." The cardinality of A is denoted by $|A|$. A set is ***infinite*** if it is not finite.
- *Theorem:*
  - Let A and B be finite sets, and suppose there is a bijection from A to B. Then $|A|=|B|$

Countability

- *Definition:*
  - A set *A* is of cardinality $\aleph_0$ denoted $|A| = \aleph_0$ if there is a bijection from *N* to *A*, where *N* is a set of all natural numbers.

- *Definition:*
  - A set *A* is ***countably infinite*** if $|A| = \aleph_0$. The set *A* is ***countable***, or ***denumerable***, if it is either finite or countably infinite. The set *A* is ***uncountable***, or ***uncountably infinite***, if it is not countable.

Cardinality

- *Definition:*
  - For any two (possibly infinite) sets A and B, we say that A and B *have the same cardinality* (written $|A|=|B|$) iff there exists a bijection (bijective function) from A to B.

Countable versus Uncountable

- *Definition:*
  - For any set S, if S is finite or $|S|=|\mathbf{N}|$, we say S is *countable*. Else, S is *uncountable.*

- Intuition behind "**countable**:" we can *enumerate* (generate in series) elements of S in such a way that *any* individual element of S will eventually be *counted* in the enumeration. Examples: **N**, **Z**.
- **Uncountable**: *No* series of elements of S (even an infinite series) can include all of S's elements. Examples: $\mathbf{R}$, $\mathbf{R}^2$, P($\mathbf{N}$)

Example of Countable Sets

- *Theorem:*
  - The set **Z** is countable.

- *Theorem:*
  - The set of all ordered pairs of natural numbers (n,m) is countable.

Example of Uncountable Sets

- ***Theorem:*** The open interval $(0,1) :\equiv \{r \in \mathbf{R} \mid 0 < r < 1\}$ is uncountable.
  - **Proof** by *diagonalization*: (Cantor, 1891)
    - Assume there is a series $\{r_i\} = r_1, r_2, \ldots$ containing *all* elements $r \in (0,1)$.
    - Consider listing the elements of $\{r_i\}$ in decimal notation (although any base will do) in order of increasing index: ... *(continued on next slide)*

Example (cont.)

A postulated enumeration of the reals:

$r_1 = 0.d_{1,1}\, d_{1,2}\, d_{1,3}\, d_{1,4}\, d_{1,5}\, d_{1,6}\, d_{1,7}\, d_{1,8} \ldots$

$r_2 = 0.d_{2,1}\, d_{2,2}\, d_{2,3}\, d_{2,4}\, d_{2,5}\, d_{2,6}\, d_{2,7}\, d_{2,8} \ldots$

$r_3 = 0.d_{3,1}\, d_{3,2}\, d_{3,3}\, d_{3,4}\, d_{3,5}\, d_{3,6}\, d_{3,7}\, d_{3,8} \ldots$

$r_4 = 0.d_{4,1}\, d_{4,2}\, d_{4,3}\, d_{4,4}\, d_{4,5}\, d_{4,6}\, d_{4,7}\, d_{4,8} \ldots$

.

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Now, consider a real number generated by taking all digits $d_{i,i}$ that lie along the *diagonal* in this figure and replacing them with *different* digits.

That real doesn't appear in the list!

Example (cont.)

- *E.g.*, a postulated enumeration of the reals:
  $r_1 =$ `0.301948571...`
  $r_2 =$ `0.103918481...`
  $r_3 =$ `0.039194193...`
  $r_4 =$ `0.918237461...`
- OK, now let's add 1 to each of the diagonal digits (mod 10), that is changing 9's to 0.
- 0.4103… can't be on the list anywhere!

Transfinite Numbers

- The cardinalities of infinite sets are not natural numbers, but are special objects called *transfinite* cardinal numbers.
- The cardinality of the natural numbers, $\aleph_0 :\equiv |\mathbf{N}|$, is the *first transfinite cardinal* number. (There are none smaller.)
- The *continuum hypothesis* claims that $|\mathbf{R}| = \aleph_1$, the *second transfinite cardinal.*

- *Proven impossible to prove or disprove!*

Countable vs. Uncountable

- You should:
  - Know how to define "same cardinality" in the case of infinite sets.
  - Know the definitions of *countable* and *uncountable*.
  - Know how to prove (at least in easy cases) that sets are either countable or uncountable.

Equipotence

- *Definition:*
  - Let A and B be sets. Then, A and B are ***equipotent*** or ***have the same cardinality***, denoted by $|A|=|B|$, if there is a bijection from A to B.

- *Theorem:*
  - Equipotence is an equivalence relation over any collection of sets.

Exercise

1. Suppose f and $f \circ g$ are one-to-one. Does it follow that g is one to one?

2. Show that the composition of functions is associative: for every three functions, *f, g, and h,* $f \circ (g \circ h) = (f \circ g) \circ h$.

3. Suppose that f is a bijective function from Y to Z and g is a bijective function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$