Discrete Mathematics

2. Relations

Binary Relations

- *Definition*:

  Let A, B be any two sets. A *binary relation* R from A to B is a subset of $A \times B$.

  E.g., Let $\mathbf{<} : \mathbf{N} \times \mathbf{N} :\equiv \{(n,m) \mid n < m\}$

- The notation $a\ R\ b$ or aRb means $(a,b) \in R$.

  E.g., $a \mathbf{<} b$ means $(a,b) \in \mathbf{<}$

- If aRb, we may say "a is related to b (by relation R)", or "a relates to b (under relation R)".

Complementary Relations

- *Definition:*

  Let $R \subseteq A \times B$ be any binary relation. Then $\overline{R}$, the *complement* of R, is the binary relation defined by

  $$\overline{R} :\equiv \{(a,b) \mid (a,b) \notin R\} = (A \times B) - R$$

- Note the complement of $\overline{R}$ is R.

Example: $\not< = \{(a,b) \mid (a,b) \notin <\} = \{(a,b) \mid \neg(a<b)\} = \geq$

Inverse Relations

- *Definition:*

  Any binary relation $R \subseteq A \times B$ has an *inverse* relation R^{-1}, defined to be
  $R^{-1} :\equiv \{(b,a) \mid (a,b) \in R\}$.

- *E.g.*, $<^{-1} = \{(b,a) \mid a<b\} = \{(b,a) \mid b>a\} = >$.
- *E.g.*, If R:People $\times$ Foods is defined by

  $aRb \Leftrightarrow$ *a eats b*, then:
  $bR^{-1}a \Leftrightarrow$ *b is eaten by a*. (Passive voice.)

Inverse Relations

- *Theorem:*
  1. $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$
  2. $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$

Relations on a Set

- *Definition:*
  - A (binary) relation from a set A to itself is called a relation *on* the set A.
  - *E.g.,* the "**<**" relation from earlier was defined as a relation *on* the set **N** of natural numbers.
  - The *identity relation* $\mathbf{I}_A$ on a set A is the set $\{(a,a) \mid a \in A\}$.

Reflexivity and Irreflexity

- *Definition:*
  - A relation R on A is *reflexive* iff (for all a in A) $(a,a) \in R$.
  - *E.g.*, the relation $\geq \equiv \{(a,b) \mid a \geq b\}$ is reflexive.
  - A relation R on A is *irreflexive* iff (for all a in A) $(a,a) \notin R$.

- Note "*irreflexive*" ≠ "*not reflexive*"!
  - *E.g.*, $<$ is irreflexive.
  - Note: "likes" between people is not reflexive, but not irreflexive either. (Not everyone likes themselves, but not everyone dislikes themselves either.)

Symmetry, Antisymmetry, and Asymmetry

- *Definition:*
  - A binary relation R on A is symmetric iff
    (for all a and b in A) $((a,b)\in R \rightarrow (b,a)\in R)$.
    - *E.g.*, = (equality) is symmetric. < is not.
    - "is married to" is symmetric, "likes" is not.
  - A binary relation R is *antisymmetric* iff
    (for all a and b in A) $(((a,b)\in R \wedge (b,a) \in R) \rightarrow (a=b))$.
  - A binary relation R is *asymmetric* iff
    (for all a and b in A) $((a,b)\in R \rightarrow (b,a)\notin R)$.

Transitivity and Intransitivity

- *Definition:*
  - A relation R is *transitive* iff (for all a, b, and c *in* A) $(((a,b)\in R \wedge (b,c)\in R) \rightarrow (a,c)\in R)$.
  - A relation is *intransitive* if it is not transitive.

- *Examples:*
  - "is an ancestor of" is transitive.
  - "likes" is intransitive.
  - "is within 1 mile of" is… ?

Totality

- *Definition:*
  - A relation $R \subseteq A \times B$ is *total* if for every a in A, there is at least one $b \in B$ such that $(a,b) \in R$.

Composite Relations

- *Definition:*
  - Let $R \subseteq A \times B$, and $S \subseteq B \times C$. Then the *composite* $R \circ S$ of R and S is defined as:
    $R \circ S = \{(a,c) \mid a \in A \wedge c \in C \wedge \exists b(b \in B \wedge (a,b) \in R \wedge (b,c) \in S)\}$

- *Definition:*
  - The n^{th} power R^n of a relation R on a set A can be defined recursively by:
    $R^0 :\equiv \mathbf{I}_A$; $R^{n+1} :\equiv R^n \circ R$ for all $n \geq 0$.

Composite Relations (cont.)

- *Theorem:*
  - $R_1 \circ (R_2 \cap R_3) \subseteq (R_1 \circ R_2) \cap (R_1 \circ R_3)$
  - $R_1 \circ (R_2 \cup R_3) = (R_1 \circ R_2) \cup (R_1 \circ R_3)$

- *Theorem:*

  Let R be a relation on a set A, i.e. $R \subseteq A \times A$, and I_A be a identity relation on a set A. ($I_A = \{\langle x,x \rangle \mid x \in A\}$)
  - R is reflexive iff $I_A \subseteq R$
  - R is irreflexive iff $I_A \cap R = \varnothing$
  - R is symmetric iff $R = R^{-1}$
  - R is asymmetric iff $R \cap R^{-1} = \varnothing$
  - R is antisymmetric iff $R \cap R^{-1} \subseteq I_A$
  - R is transitive iff $R \circ R \subseteq R$

Representing relations using digraphs : Walk, loop, sling(1/2)

- *Definition:*
  - A ***walk*** is a sequence $x_0, x_1, \ldots, x_n$ of the vertices of a digraph such that $x_i x_{i+1}$, $0 \le i \le n-1$, is an edge.
  - The ***length of a walk*** is the number of edges in the walk.
  - If a walk holds $x_i \neq x_j$ $(i \neq j)$ $i,j=0, \ldots, n$, except x_0, x_n(i.e $x_0=x_n$), the walk is called a ***cycle.***
  - A ***loop*** is a cycle of length one.
  - A ***sling*** is a cycle of length two.

Representing relations using digraphs : Walk, loop, sling(2/2)

- *Theorem:*

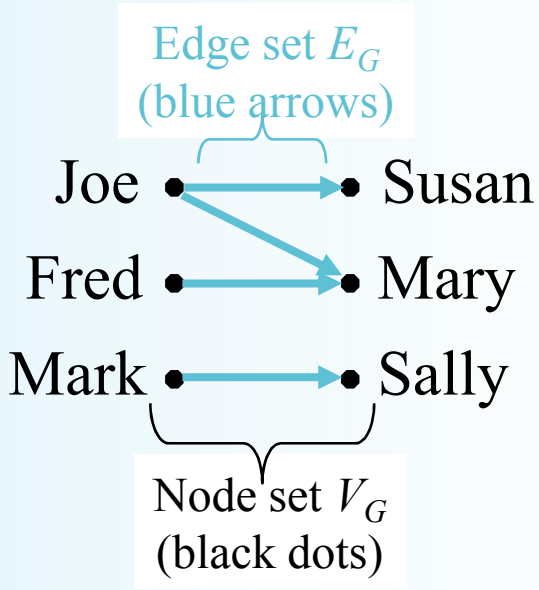
  Given a digraph G=<N, V> where N is a set of nodes and V is a set of edges,

  1. *R* is reflexive iff *G* has a loop at every node.
  2. *R* is irreflexive iff *G* has no loop at any node.
  3. *R* is symmetric iff if *G* has a walk of length one between two distinct nodes, then it has a sling between them.
  4. *R* is asymmetric iff if *G* has a walk of length one between two distinct nodes, then it has no sling between them and no loop at any node.
  5. *R* is antisymmetric iff if *G* has a walk of length one between two distinct nodes, then it has no sling between them.
  6. *R* is transitive iff if *G* has a walk of length two between two nodes, then it has a walk of length one between them

Using Directed Graphs

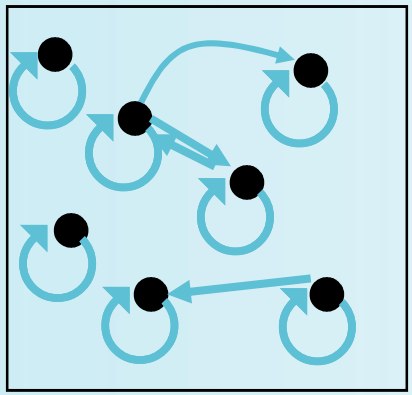
- A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs,links)*. Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R \subseteq A \times B$ can be represented as a graph $G_R=(V_G=A \cup B, E_G=R)$.

	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

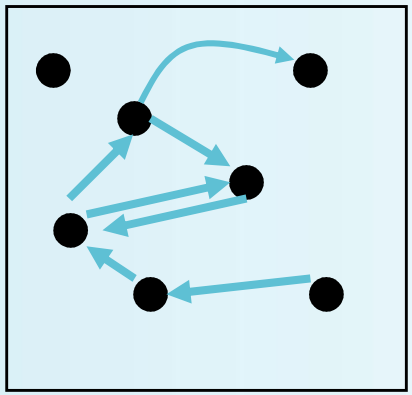


Digraph Reflexive, Symmetric

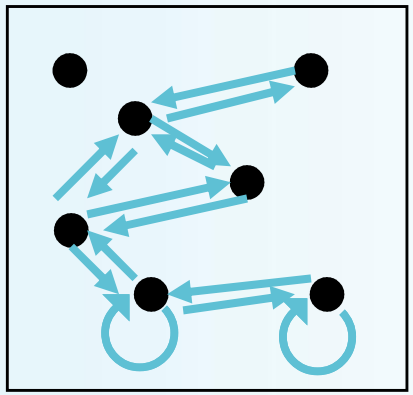
It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



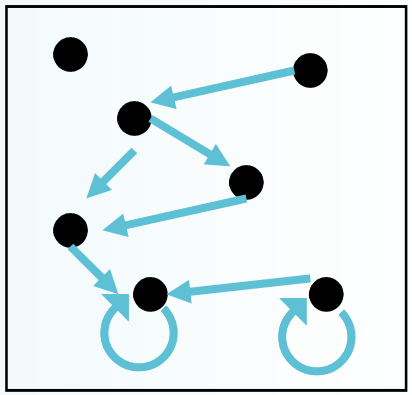
Reflexive: Every node has a loop



Irreflexive: No node has a loop



Symmetric: Every link is bidirectional



Antisymmetric: No link is bidirectional

Closures of Relations

- *Definition:*

  For any property X, the "X closure" of a set R is defined as the "smallest" superset of R that has the given property.

- *Theorem:*
  - The *reflexive closure* of a relation R on A is obtained by adding (a,a) to R for each $a \in A$. *I.e.*, it is $\boldsymbol{R} \cup \boldsymbol{I}_A$
  - The *symmetric closure* of R is obtained by adding (b,a) to R for each (a,b) in R. *I.e.*, it is $\boldsymbol{R} \cup \boldsymbol{R}^{-1}$
  - The *transitive closure* or *connectivity relation* of R is obtained by repeatedly adding (a,c) to R for each $(a,b),(b,c)$ in R.
    - *I.e.*, it is $R^* = \bigcup_{n \in \mathbf{Z}^+} R^n$

Equivalence Relations

- *Definition :*
  - A relation R on a set A is called an *equivalence relation* iff it is reflexive, symmetric, and transitive

- Examples :
  - "Strings *a* and *b* are the same length."
  - "Integers *a* and *b* have the same absolute value."
  - "Positive real numbers *a* and *b* have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$)."
  - "Integers *a* and *b* have the same residue modulo *m*." (for a given $m>1$)

Equivalence Classes

- *Definition :*
  - Let R be any equivalence relation on a set A. For each a in A, the *equivalence class* of a with respect to R, denoted by $[a]_{R,}$ is

$$[a]_R = \{ s \mid \langle a,s \rangle \in R\} = a/R$$

Equivalence Classes

- Examples :
  - "Strings a and b are the same length."
    - $[a]_R$ = the set of all strings of the same length as a.
  - "Integers a and b have the same absolute value."
    - $[a]_R$ = the set $\{a, -a\}$
  - "Positive real numbers a and b have the same fractional part (*i.e.*, $a - b \in \mathbf{Z}$)."
    - $[a]_R$ = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$
  - "Integers a and b have the same residue modulo m." (for a given $m>1$)
    - $[a]_R$ = the set $\{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$

Some theorems

- *Theorem:*
  - Let R be an equivalence relation on a set A. Then ,
    1. $x \in [x]_R$, for all x in A.
    2. if $\langle x, y \rangle \in R$, then $[x]_R = [y]_R$

- *Theorem:*

  Let R be an equivalence relation on a set A. Then,

  if $\langle x, y \rangle \notin R$ then $[x]_R \cap [y]_R = \varnothing$

Partition and Covering of a Set

- *Definition:*
  - Let S be a given set and $A = \{A_1, A_2, \ldots, A_m\}$ where each A_i, i=1, … m, is a non-empty subset of S and $\bigcup_{i=1}^{m} A_i = S$.
  - Then the set A is called a ***covering*** of S, and the sets $A_1, A_2, \ldots, A_m$ are said to ***cover*** S.
  - If the elements of A, which are subsets of S, are mutually disjoint, then A is called a ***partition*** of S, and the sets $A_1, A_2, \ldots, A_m$ are called the ***blocks*** of the partition.

Quotient set

- *Definition:*
  - Let R be an equivalence relation on a set A, then $A/R = \{[x]_R \mid x \in A\}$ is called a ***quotient set of A modulo R.***

- *Theorem:*
  - Let R be an equivalence relation on a set A, then the quotient set of A modulo R is a partition of A.

Relation induced by the partition

- *Definition*:
  - Let A be a set. Let $\Pi = \{A_1, A_2, \ldots, A_n\}$ be a partition of A. R_Π is a ***relation induced by the partition*** Π and defined as follows.

    $R_\Pi = \{\langle x, y\rangle \mid (x \in A_i) \wedge (y \in A_i), \text{ for some } i\}$

- *Theorem*:
  - Let A be a set. Let $\Pi = \{A_1, A_2, \ldots, A_n\}$ be a partition A and R_Π be the relation induced by the partition Π. Then, R_Π is an equivalence relation on A.

Refinement

- *Definition*:
  - Let Π_1 and Π_2 be two partitions of a set A. Π_2 is a ***refinement*** of Π_1, (Π_2 refines Π_1), if for every block B_i in Π_2, there exists some block A_j in Π_1 such that $B_i \subseteq A_j$.

- *Theorem*:
  - Let Π and Π' be two partitions of a nonempty set A and let R_Π and $R_{\Pi'}$ be the equivalence relations induced by and Π and Π' respectively. Then Π' refines Π if and only if $R_{\Pi'} \subseteq R_\Pi$.

Partial Orderings

- *Definition*:
  - A relation R on a set S is called a **partial ordering** or **partial order** iff it is reflexive, antisymmetric, and transitive.
  - A set S together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by (S, R).

Partial Orderings (cont.)

- *Example*:

  Consider the "greater than or equal to" relation ≥ (defined by $\{(a, b) \mid a \geq b\}$). Is ≥ a **partial ordering** on the set of integers?

  proof:

  1. ≥ is **reflexive**, because $a \geq a$ for every integer a.
  2. ≥ is **antisymmetric**, because if $a \geq b \wedge b \geq a$, then $a = b$.
  3. ≥ is **transitive**, because if $a \geq b$ and $b \geq c$, then $a \geq c$.

  Consequently, $(Z, \geq)$ is a partially ordered set.

Partial Orderings (cont.)

- *Example*:

  Is the "inclusion relation" $\subseteq$ a **partial ordering** on the power set of a set S?

  Proof:

  1. $\subseteq$ is **reflexive**, because $A \subseteq A$ for every set A.
  2. $\subseteq$ is **antisymmetric**, because if $A \subseteq B \wedge B \subseteq A$, then $A = B$.
  3. $\subseteq$ is **transitive**, because if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

  Consequently, $(\wp(S), \subseteq)$ is a partially ordered set.

Partial Orderings (cont.)

- In a poset the notation $a \leq b$ denotes that $(a, b) \in R$. Note that the symbol $\leq$ is used to denote the relation in **any** poset, not just the "less than or equal" relation. The notation $a < b$ denotes that $a \leq b$, but $a \neq b$. If $a < b$ we say "a is less than b" or "b is greater than a".
- For two elements a and b of a poset $(S, \leq)$ it is possible that neither $a \leq b$ nor $b \leq a$. For instance, in $(\wp(Z), \subseteq)$, {1, 2} is not related to {1, 3}, and vice versa, since neither is contained within the other.

Partial Orderings (cont.)

- *Definition*:
  - The elements a and b of a poset $(S, \leq)$ are called **comparable** if either $a \leq b$ or $b \leq a$.
    When a and b are elements of S such that neither $a \leq b$ nor $b \leq a$, then a and b are called **incomparable**.

Partial Orderings (cont.)

- For some applications, we require **all** elements of a set to be comparable.
  - For example, if we want to write a dictionary, we need to define an order on **all** English words (alphabetic order).

- *Definition* **:**
  - If $(S, \leq)$ is a poset and **every two elements** of S are comparable, $(S, \leq)$ is called a **totally ordered** or **linearly ordered set**, and $\leq$ is called a **total order** or **linear order**. A totally ordered set is also called a **chain.**

Partial Orderings (cont.)

- *Example I:* Is $(Z, \leq)$ a totally ordered poset?

  Yes, because $a \leq b$ or $b \leq a$ for all integers a and b.

- *Example II:* Is $(Z^+, |)$ a totally ordered poset?

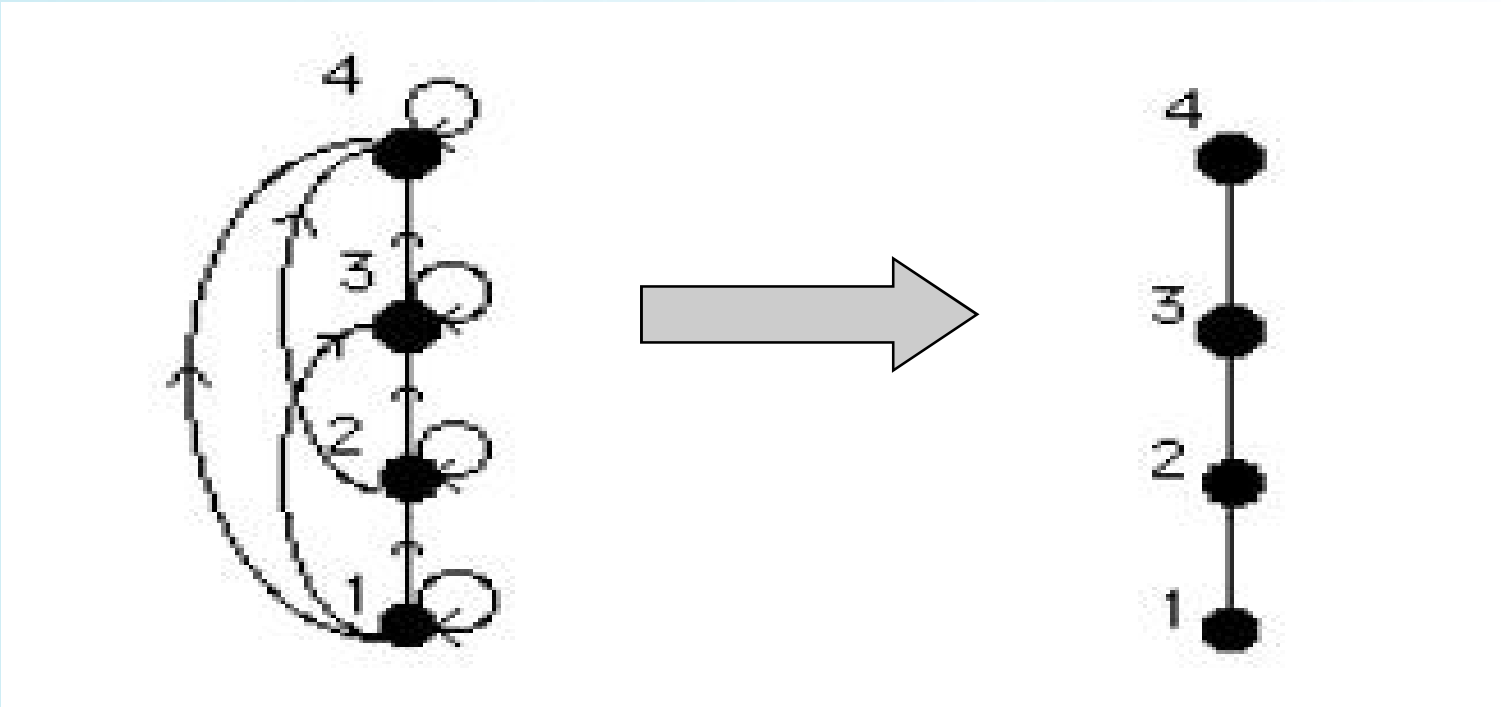
  No, because it contains incomparable elements such as 5 and 7.

Hasse Diagram

- *Definition :*
  Let G be a digraph representing a poset, (A, ≤).
  The Hasse diagram of (A, ≤) is constructed from G by
  1. All loops are omitted.
  2. An arc is not present in a Hasse diagram if it is implied by the transitivity of the relation.
  3. All arcs point upward and arrow heads are not used.

Example of Hasse Diagram

- { <a, b> | a<=b } on {1, 2, 3, 4}



Greatest and Least elements

- *Definition:*

  Let $< A, \leq >$ be a poset and B be a subset of A.

  1. An element a $\in B$ is ***a greatest element*** of B iff for every element a'$\in B$, a' $\leq$a.
  2. An element a $\in B$ is ***a least element*** of B iff for every element a'$\in B$, a $\leq$a '.

- *Theorem:*
  - Let $< A, \leq >$ be a poset and $B \subseteq A$. If a and b are greatest(least) elements of B, then $a=b$

Least upper bound (lub)

- *Definition*:

  Let $< A, \leq >$ be a poset and B be a subset of A.

  1. An element $a \in A$ is an ***upper bound*** for B iff for every element $a' \in B$, $a' \leq a$.
  2. An element $a \in A$ is a ***least upper bound (lub)*** for B iff a is an upper bound for B and for every upper bound a' for B, $a \leq a'$.

Greatest lower bound (glb)

- *Definition:*

  Let $< A, \leq >$ be a poset and B be a subset of A.

  1. An element $a \in A$ is a ***lower bound*** for B iff for every element $a' \in B$, $a \leq a'$.
  2. An element $a \in A$ is a ***greatest lower bound (glb)*** for B iff a is a lower bound for B and for every lower bound a' for B, $a' \leq a$.

lub and glb

- *Theorem:*

  Let $< A, \leq >$ be a poset and $B \subseteq A$.

  1. If b is a greatest element of B, then b is a lub of B.
  2. If b is an upper bound of B and $b \in B$, then b is a greatest element of B.

- *Theorem:*

  - Let $< A, \leq >$ be a poset and $B \subseteq A$.

  If a least upper bound (a greatest lower bound)for B exists, then it is unique.

Lattices

- *Definition*:
  - A poset is a ***lattice*** if every pair of elements has a lub and a glb.

- *Theorem*:

  Let <L, ≤> be a lattice. If $x*y$ ($x+y$) denotes the glb (lub) for $\{x,y\}$, then the following holds.

  For any $a, b, c \in L$,

*(i) a*a=a*	*(i') a+a=a*	*(Idempotent)*
*(ii) a*b=b*a*	*(ii') a+b=b+a*	*(Commutative)*
*(iii) (a*b)*c= a*(b*c)*	*(iii') (a+b)+c= a+(b+c)*	*(Associative)*
(iv) a(a+b)=a*	*(iv') a+(a*b)=a*	*(Absorption)*

Exercise

1. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if $x+y=0$

2. Suppose that R and S are reflexive relations on a set A. Prove or disprove each of theses statements

   (a) $R \cup S$ is reflexive

   (b) $R \cap S$ is reflexive

3. Show that the relation R on a set A is symmetric if and only if $R=R^{-1}$, where R^{-1} is the inverse relation.

Exercise (cont.)

4. Let A be a set of ordered pairs of positive integers and R be a relation on A such that $\langle(x,y), (u,v)\rangle \in R$ if and only if $x+v = y+u$. Determine whether or not R is an equivalence relation.

5. Let $\langle A, \leq \rangle$ be a lattice. Prove that for every x, y, and z in A,

   (a) $x*(y*z) = (x*y)*z$

   (b) $x+(x*y) = x$

   where $x*y$ is glb(x,y) and $x+y$ is lub(x,y).

Exercise (cont.)

6. (a) Draw the Hasse diagram for the p.o.set <P(A), $\subseteq$> where A = {a,b,c} and $\subseteq$ means "is a subset of."

(b) For the set B={{a}, {b}, {c}}, determine the followings:

i. The least element

ii. The greatest element

iii. The lower bound

iv. The upper bound

v. The glb (greatest lower bound)

vi. The lub (least upper bound)