Discrete Mathematics

1-3. Set Operations

Introduction to Set Theory

- A *set* is a new type of structure, representing an *unordered* collection (group, plurality) of zero or more *distinct* (different) objects.
- Set theory deals with operations between, relations among, and statements about sets.
- Sets are ubiquitous in computer software systems.
- *All* of mathematics can be defined in terms of some form of set theory (using predicate logic).

Basic notations for sets

- For sets, we'll use variables *S*, *T*, *U*, …
- We can denote a set *S* in writing by listing all of its elements in curly braces:
  - {a, b, c} is the set of whatever 3 objects are denoted by a, b, c.
- *Set builder notation*: For any proposition $P(x)$ over any universe of discourse, $\{x \mid P(x)\}$ is *the set of all x such that P(x)*.

Basic properties of sets

- Sets are inherently *unordered*:
  - No matter what objects a, b, and c denote, {a, b, c} = {a, c, b} = {b, a, c} = {b, c, a} = {c, a, b} = {c, b, a}.
- All elements are *distinct* (unequal); multiple listings make no difference!
  - If a=b, then {a, b, c} = {a, c} = {b, c} = {a, a, b, a, b, c, c, c, c}.
  - This set contains at most 2 elements!

Infinite Sets

- Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).
- Symbols for some special infinite sets:
  N = {0, 1, 2, …} The **N**atural numbers.
  Z = {…, -2, -1, 0, 1, 2, …} The *Z*ntegers.
  R = The "**R**eal" numbers, such as 374.1828471929498181917281943125…
- Infinite sets come in different sizes!

Empty Set

- *Definition:*

  A set which does not contain any elements is an ***empty set****, denoted by* $\varnothing$ *or* $\{\}$ *or* $\{x \mid \textbf{\textit{false}}\}$

- Example: $\neg\exists x: x \in \varnothing$

Subset and Superset

- *Definition:*

  *Let S and T be any two sets. S is a **subset** of T and T is a **superset** of S, denoted by $S \subseteq T$, if and only if* every element of S is an element of T,

  i.e., $(\forall x)\ ((x \in S) \rightarrow (x \in T))$.

- Example

  $\emptyset \subseteq S$, $S \subseteq S$.

Set Equality

- *Definition:*

  Let S and T be any two sets. S and T are declared to be equal *if and only if* they contain exactly the same elements, i.e. $S=T$ iff $(S \subseteq T) \wedge (S \subseteq T)$

- Note that it does not matter *how the set is defined or denoted.*

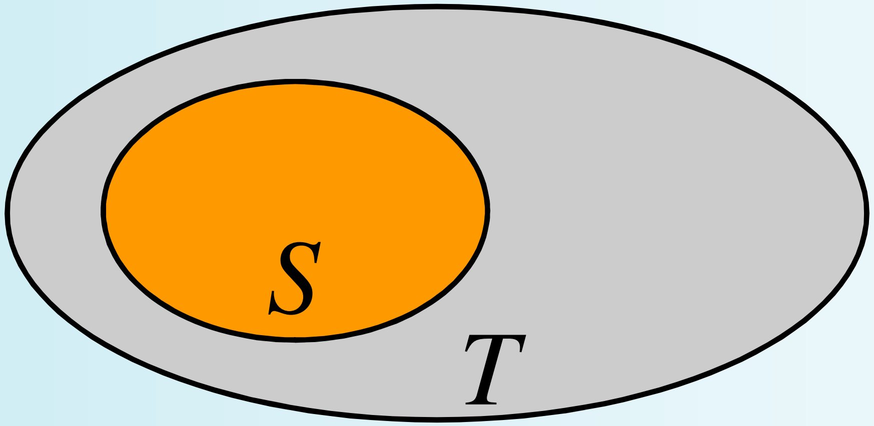
  Example:

  The set {1, 2, 3, 4} =
  $\{x \mid x$ is an integer where $x>0$ and $x<5\}$ =
  $\{x \mid x$ is a positive integer whose square is >0 and $<25\}$

Proper Subsets & Supersets

- *Definition:*

  Let S and T be any two sets. S is a proper subset of T (T is a proper superset of S), denoted by $S \subset T$ *iff* $S \subseteq T$ and $S \neq T$.



Venn Diagram equivalent of $S \subset T$

Example:
$\{1,2\} \subset \{1,2,3\}$

Sets Are Objects, Too!

- The objects that are elements of a set may *themselves* be sets.
- Example:

  Let $S=\{x \mid x \subseteq \{1,2,3\}\}$
  then $S=\{\emptyset,$
  $\{1\}, \{2\}, \{3\},$
  $\{1,2\}, \{1,3\}, \{2,3\},$
  $\{1,2,3\}\}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$!!!!

Basic Set Relations: Member of

- *Definition:*
  - $x \in S$ ("x is in S") is the proposition that object x is an $\in$*lement* or *member* of set S.
    - *e.g.* $3 \in \mathbf{N}$, "a"$\in \{x \mid x$ is a letter of the alphabet$\}$
  - Can define set equality in terms of $\in$ relation:
    $\forall S,T$: $S=T \leftrightarrow (\forall x$: $x \in S \leftrightarrow x \in T)$
    "Two sets are equal iff they have all the same members."

- $x \notin S :\equiv \neg(x \in S)$ "x is not in S"

Cardinality and Finiteness

- $|S|$ (read "the *cardinality* of *S*") is a measure of how many different elements *S* has.
- *E.g.*, $|\varnothing|=0$, $|\{1,2,3\}| = 3$, $|\{a,b\}| = 2$,
  $|\{\{1,2,3\},\{4,5\}\}| = 2$
- If $|S| \in \mathbf{N}$, then we say *S* is *finite*. Otherwise, we say *S* is *infinite*.
- What are some infinite sets we've seen?

Power Set

- *Definition:*

  Let S be a set. The *power set* $\wp(S)$ of a set S is the set of all subsets of S. $\wp(S) = \{x \mid x \subseteq S\}$.

- Example:

  $\wp(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.

- Sometimes $\wp(S)$ is written $\mathbf{2}^S$.
  Note that for finite S, $|\wp(S)| = 2^{|S|}$.

- It turns out that $|P(\mathbf{N})| > |\mathbf{N}|$.
  There are different sizes of infinite sets!

Ordered *n*-tuples

- *Definition:*

  For $n \in \mathbf{N}$, an *ordered n-tuple* or a *sequence of length n* is written $(a_1, a_2, \ldots, a_n)$. The *first* element is a_1, *etc.*

- These are like sets, except that duplicates matter, and the order makes a difference.
- Note $(1, 2) \neq (2, 1) \neq (2, 1, 1)$.
- Empty sequence, singlets, pairs, triples, quadruples, quin<u>tuples</u>, …, *n*-tuples.

Cartesian Products of Sets

- *Definition:*

  Let A and B be any two sets.

  The *Cartesian product* $A \times B$ is defined to be
  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

- Example: $\{a,b\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2)\}$
- Note that for finite A, B, $|A \times B| = |A||B|$.
- Note that the Cartesian product is *not* commutative: $A \times B \neq B \times A$.
- Extends to $A_1 \times A_2 \times \ldots \times A_n$...

Union Operator

- *Definition:*

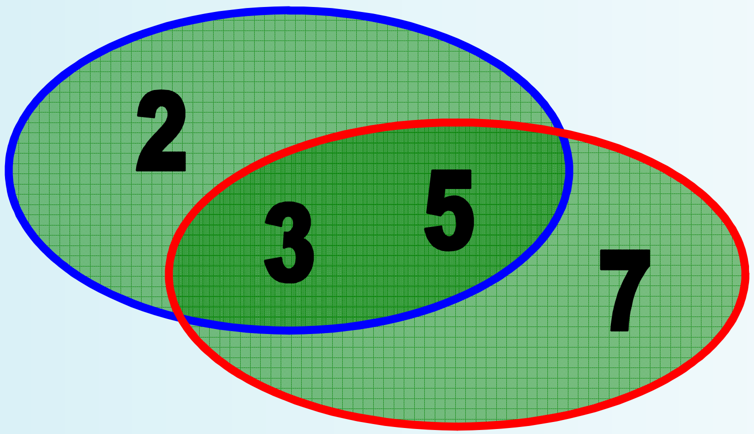
  Let A and B be any two sets. The $\cup$*nion* $A\cup B$ of A and B is the set containing all elements that are either in A **or** (“$\vee$**”)** in B (or, of course, in both), i.e.,

  $A\cup B = \{x \mid x\in A \vee x\in B\}$.

- Note that $A\cup B$ contains all the elements of A **and** it contains all the elements of B: $(A\cup B \supseteq A) \wedge (A\cup B \supseteq B)$

Example of Union

- $\{a,b,c\} \cup \{2,3\} = \{a,b,c,2,3\}$
- $\{2,3,5\} \cup \{3,5,7\} = \{2,3,5,3,5,7\} = \{2,3,5,7\}$



Intersection Operator

- *Definition:*

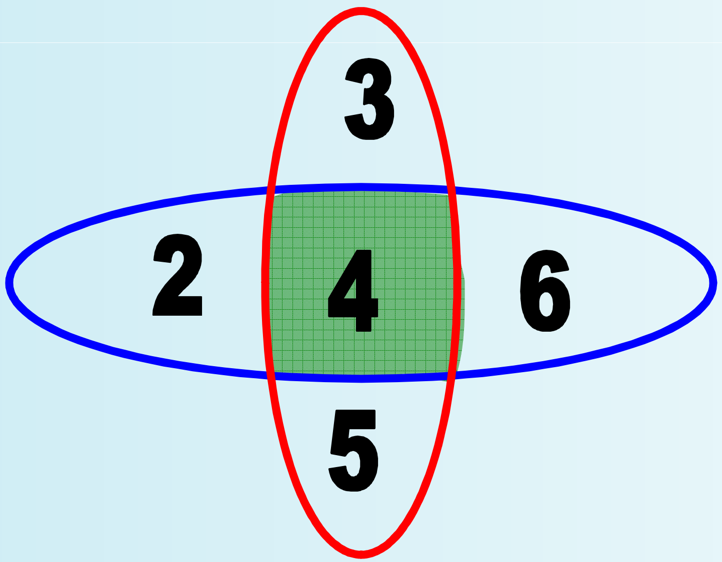
  Let A and B be any two sets. The *intersection* $A \cap B$ of A and B is the set containing all elements that are simultaneously in A **and** ("$\wedge$") in B, *i.e.*,

  $A \cap B \equiv \{x \mid x \in A \wedge x \in B\}$.

- Note that $A \cap B$ is a subset of A **and** it is a subset of B: $(A \cap B \subseteq A) \wedge (A \cap B \subseteq B)$

Example of Intersection

- $\{a,b,c\} \cap \{2,3\} = \varnothing$
- $\{2,4,6\} \cap \{3,4,5\} = \{4\}$



Disjointedness

- *Definition:*

  Let A and B be any two sets. A and B are called *disjoint* (*i.e.*, unjoined)
  iff their intersection is empty ($A \cap B = \emptyset$).

- Example:

  The set of even integers is disjoint with the set of odd integers.

Inclusion-Exclusion Principle

- How many elements are in $A \cup B$?

  $|A \cup B| = |A| + |B| - |A \cap B|$
- Example:

  How many students are on our class email list?
  Consider set $E = I \cup M$,
  $I = \{s \mid s$ turned in an information sheet$\}$
  $M = \{s \mid s$ sent the TAs their email address$\}$
- Some students did both!

  $|E| = |I \cup M| = |I| + |M| - |I \cap M|$

Set Difference

- *Definition:*

  Let A and B be any two sets. The set *difference of A and B*, $A-B$ is the set of all elements that are in A but not B.

  $$A - B = \{x \mid x \in A \wedge x \notin B\}$$
  $$= \{x \mid \neg(x \in A \rightarrow x \in B)\}$$

  $A - B$ is also called the *complement of B with respect to A*.

Example of Set Difference

- $\{1,2,3,4,5,6\} - \{2,3,5,7,9,11\} =$ **$\{1,4,6\}$**
- $\mathbf{Z} - \mathbf{N} = \{\dots, -1, 0, 1, 2, \dots\} - \{0, 1, \dots\}$
  $= \{x \mid x \text{ is an integer but not a nat. \#}\}$
  $= \{x \mid x \text{ is a negative integer}\}$
  $= \{\dots, -3, -2, -1\}$

Universal Set & Complement of a Set

- *Definition* (Universal Set):

  A set is a **universal set** or a **universe of discourse**, denoted by U, if it includes every set under discussion.

- *Definition* (Complement of a Set):

  Let A be a set. The ***complement*** of A in U, denoted by $\overline{A}$, is the set of all elements of U which are not elements of A, i.e.,

  $\overline{A} = U - A$.

  $U - A = \{x \mid x \in U \land x \notin A\}$

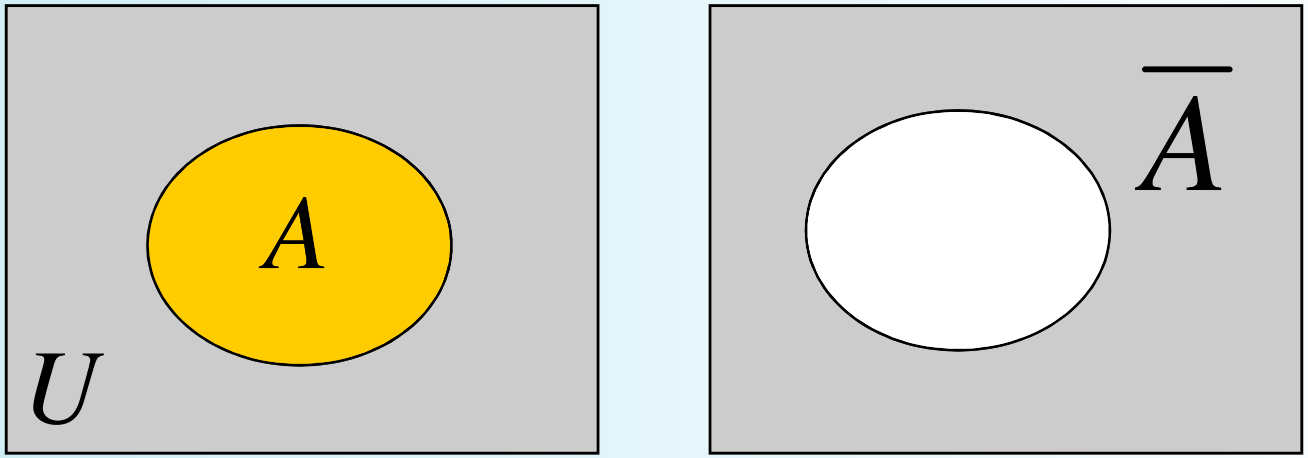
- Example:

  If U=**N**, $\overline{\{3,5\}} = \{1,2,4,6,7,\ldots\}$

More on Set Complements

- An equivalent definition, when *U* is clear:

$$\overline{A} = \{x \mid x \notin A\}$$



Set Identities (Theorem)

- *Theorem*:
  - Identity: $A \cup \emptyset = A \quad A \cap U = A$
  - Domination: $A \cup U = U \quad A \cap \emptyset = \emptyset$
  - Idempotent: $A \cup A = A = A \cap A$
  - Double complement: $\overline{(\overline{A})} = A$
  - Commutative: $A \cup B = B \cup A \quad A \cap B = B \cap A$
  - Associative: $A \cup (B \cup C) = (A \cup B) \cup C$
    $A \cap (B \cap C) = (A \cap B) \cap C$

DeMorgan's Law for Sets

- *Theorem*:
  - Exactly analogous to (and derivable from) DeMorgan's Law for propositions.

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Proving Set Equality

To prove statements about the form $E_1 = E_2$ where E_1 and E_2 are sets, prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.

Example: Show $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

- Show $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
  - Assume $x\in A\cap(B\cup C)$, & show $x\in (A\cap B)\cup(A\cap C)$.
  - We know that $x\in A$, and either $x\in B$ or $x\in C$.
    - Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in (A\cap B)\cup(A\cap C)$.
    - Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in (A\cap B)\cup(A\cap C)$.
  - Therefore, $x\in (A\cap B)\cup(A\cap C)$.
  - Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
- Show $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$. …

Theorem:

- If *A* and *B* are two sets, the following statements are equivalent.

(1) $A \subseteq B$

(2) $A \cap B = A$

(3) $A \cup B = B$

Generalized Unions & Intersections

- Since union & intersection are commutative and associative, we can extend them from operating on *ordered pairs* of sets (A,B) to operating on sequences of sets $(A_1,\ldots,A_n)$, or even unordered *sets* of sets, $X=\{A \mid Q(A)\}$.

Generalized Union

- Binary union operator: $A \cup B$
- n-ary union:
  $A_1 \cup A_2 \cup \ldots \cup A_n :\equiv ((\ldots((A_1 \cup A_2) \cup \ldots) \cup A_n)$
  (grouping & order is irrelevant)
- "Big U" notation: $\bigcup_{i=1}^{n} A_i$
- Or for infinite sets of sets: $\bigcup_{A \in X} A$

Generalized Intersection

- Binary intersection operator: $A \cap B$
- n-ary intersection:
  $A \cap A_2 \cap \ldots \cap A_n \equiv ((\ldots((A_1 \cap A_2) \cap \ldots) \cap A_n)$
  (grouping & order is irrelevant)
- "Big Arch" notation: $\bigcap_{i=1}^{n} A_i$
- Or for infinite sets of sets: $\bigcap_{A \in X} A$

Exercise

1. Let A and B be sets. Show that

   (a) $(A \cap B) \subseteq A$

   (b) $A \cup (B - A) = A \cup B$

2. Let A, B and C be sets. Show that

   $(A - B) - C = (A - C) - (B - C)$.

3. Let A and B be two sets. Prove or disprove each of the followings

   (a) $P(A) \cup P(B) \subseteq P(A \cup B)$ where $P(A)$ is the power set of the set A.

   (b) $P(A \cup B) \subseteq P(A) \cup P(B)$