Discrete Mathematics

1-1. Logic

Foundations of Logic

Mathematical Logic is a tool for working with complicated *compound* statements. It includes:

- A language for expressing them.
- A concise notation for writing them.
- A methodology for objectively reasoning about their truth or falsity.
- It is the foundation for expressing formal proofs in all branches of mathematics.

Propositional Logic

Propositional Logic is the logic of compound statements built from simpler statements using so-called *Boolean connectives.*

Some applications in computer science:

- Design of digital electronic circuits.
- Expressing conditions in programs.
- Queries to databases & search engines.

Definition of a *Proposition*

A *proposition* is simply a *statement* (*i.e.*, a declarative sentence) *with a definite meaning*, having a *truth value* that's either *true* (T) or *false* (F) (**never** both, neither, or somewhere in between).

A *proposition* (*statement*) may be denoted by a variable like *P*, *Q*, *R*,…, called a *proposition (statement) variable*.

Examples of Propositions

- "It is raining." (In a given situation.)
- "Beijing is the capital of China."
- "1 + 2 = 3"

But, the following are **NOT** propositions:

- "Who's there?" (interrogative, question)
- "La la la la la." (meaningless interjection)
- "Just do it!" (imperative, command)
- "1 + 2" (expression with a non-true/false value)

Operators / Connectives

An *operator* or *connective* combines one or more *operand* expressions into a larger expression. (*E.g.*, "+" in numeric exprs.)

Unary operators take 1 operand (*e.g.*, −3)

binary operators take 2 operands (*e.g.*, 3×4).

Propositional or *Boolean* operators operate on propositions or truth values instead of on numbers.

Some Popular Boolean Operators

Formal Name	**Nickname**	**Arity**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

The Negation Operator

The unary *negation operator* "¬" (*NOT*) transforms a prop. into its logical *negation*.

E.g. If p = "I have brown hair."

then $\neg p$ = "I do **not** have brown hair."

Truth table for NOT:

p	$\neg p$
T	F
F	T

Operand column — Result column

T :≡ True; F :≡ False

":≡" means "is defined as"

The Conjunction Operator

The binary *conjunction operator* "$\wedge$" (*AND*) combines two propositions to form their logical *conjunction*.

E.g. If p="I will have salad for lunch." and q="I will have steak for dinner.", then $p \wedge q$="I will have salad for lunch **and** I will have steak for dinner."

Remember: "$\wedge$" points up like an "A", and it means "ΛND"

Conjunction Truth Table

- Note that a conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of n propositions will have 2^n rows in its truth table.

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

- Also: $\neg$ and $\wedge$ operations together are sufficient to express *any* Boolean truth table!

The Disjunction Operator

The binary *disjunction operator* "$\lor$" (*OR*) combines two propositions to form their logical *disjunction*.

p="My car has a bad engine."

q="My car has a bad carburetor."

$p \lor q$="Either my car has a bad engine, **or** my car has a bad carburetor."

Meaning is like "and/or" in English.

After the downward-pointing "axe" of "$\lor$" splits the wood, you can take 1 piece OR the other, or both.

Disjunction Truth Table

- Note that $p \lor q$ means that p is true, or q is true, **or both** are true!
- So, this operation is also called *inclusive or,* because it **includes** the possibility that both p and q are true.
- "¬" and "∨" together are also universal (sufficient to express *any* Boolean truth table).

p	q	$p \lor q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

Note difference from AND

Nested Propositional Expressions

- Use parentheses to *group sub-expressions*:
  "I just saw my old *f*riend, and either he's *g*rown or I've *s*hrunk." = $f \wedge (g \vee s)$
  - $(f \wedge g) \vee s$ would mean something different
  - $f \wedge g \vee s$ would be ambiguous
- By convention, "¬" takes *precedence* over both "∧" and "∨".
  - $\neg s \wedge f$ means $(\neg s) \wedge f$, **not** $\neg (s \wedge f)$

A Simple Exercise

Let p="It rained last night",
q="The sprinklers came on last night,"
r="The lawn was wet this morning."

Translate each of the following into English:

$\neg p$ = "It didn't rain last night."

$r \wedge \neg p$ = "The lawn was wet this morning, and it didn't rain last night."

$\neg r \vee p \vee q$ = "Either the lawn wasn't wet this morning, or it rained last night, or the sprinklers came on last night."

The *Exclusive Or* Operator

The binary *exclusive-or operator* "$\oplus$" (*XOR*) combines two propositions to form their logical "exclusive or" (exjunction?).

p = "I will earn an A in this course,"

q = "I will drop this course,"

$p \oplus q$ = "I will either earn an A in this course, or I will drop it (but not both!)"

Exclusive-Or Truth Table

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

Note difference from OR.

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called *exclusive or,* because it **excludes** the possibility that both p and q are true.
- "¬" and "⊕" together are **not** universal.

Natural Language is Ambiguous

Note that <u>English</u> "or" can be ambiguous regarding the "both" case!

"Pat is a singer or Pat is a writer." – $\vee$

"Pat is a man or Pat is a woman." – $\oplus$

p	q	p "or" q
F	F	F
F	T	T
T	F	T
T	T	?

Need context to disambiguate the meaning!

For this class, assume "or" means <u>inclusive</u>.

The *Implication* Operator

antecedent consequent

The *implication* $p \to q$ states that p implies q.

I.e., If p is true, then q is true; but if p is not true, then q could be either true or false.

E.g., let p = "You study hard."
q = "You will get a good grade."

$p \to q$ = "If you study hard, then you will get a good grade." (else, it could go either way)

Implication Truth Table

- $p \rightarrow q$ is **false** only when p is true but q is **not** true.
- $p \rightarrow q$ does **not** say that p causes q!
- $p \rightarrow q$ does **not** require that p or q **are ever true**!
- *E.g.* "(1=0) $\rightarrow$ pigs can fly" is TRUE!

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	**F**
T	T	T

The only False case!

Examples of Implications

- "If this lecture ends, then the sun will rise tomorrow." *True* or *False*?
- "If Tuesday is a day of the week, then I am a penguin." *True* or *False*?
- "If 1+1=6, then Bush is president." *True* or *False*?
- "If the moon is made of green cheese, then I am richer than Bill Gates." *True* or *False*?

Why does this seem wrong?

- Consider a sentence like,
  - “If I wear a red shirt tomorrow, then the U.S. will attack Iraq the same day.”
- In logic, we consider the sentence **True** so long as either I don’t wear a red shirt, or the US attacks.
- But in normal English conversation, if I were to make this claim, you would think I was lying.
  - Why this discrepancy between logic & language?

Resolving the Discrepancy

- In English, a sentence "if *p* then *q*" usually really *implicitly* means something like,
  - "In all possible situations, if *p* then *q*."
    - That is, "For *p* to be true and *q* false is *impossible*."
    - Or, "I *guarantee* that no matter what, if *p*, then *q*."
- This can be expressed in *predicate logic* as:
  - "For all situations *s*, if *p* is true in situation *s*, then *q* is also true in situation *s*"
  - Formally, we could write: $\forall s, P(s) \rightarrow Q(s)$
- This sentence is logically ***False*** in our example, because for me to wear a red shirt and the U.S. *not* to attack Iraq is a *possible* (even if not actual) situation.
  - Natural language and logic then agree with each other.

English Phrases Meaning $p \to q$

- "p implies q"
- "if p, then q"
- "if p, q"
- "when p, q"
- "whenever p, q"
- "q if p"
- "q when p"
- "q whenever p"
- "p only if q"
- "p is sufficient for q"
- "q is necessary for p"
- "q follows from p"
- "q is implied by p"

We will see some equivalent logic expressions later.

Converse, Inverse, Contrapositive

Some terminology, for an implication $p \rightarrow q$:

- Its *converse* is: $q \rightarrow p$.
- Its *inverse* is: $\neg p \rightarrow \neg q$.
- Its *contrapositive*: $\neg q \rightarrow \neg p$.
- One of these three has the *same meaning* (same truth table) as $p \rightarrow q$. Can you figure out which?

How do we know for sure?

Proving the equivalence of $p \rightarrow q$ and its contrapositive using truth tables:

p		q	$\neg q$		$\neg p$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
F	$\rightarrow$	F	T	$\rightarrow$	T	T	T
F	$\rightarrow$	T	F	$\rightarrow$	T	T	T
T	$\not\rightarrow$	F	T	$\not\rightarrow$	F	F	F
T	$\rightarrow$	T	F	$\rightarrow$	F	T	T

The *biconditional* operator

The *biconditional* $p \leftrightarrow q$ states that p is true *if and only if (IFF)* q is true.

p = "You can take the flight."

q = "You buy a ticket"

$p \leftrightarrow q$ = "You can take the flight if and only if you buy a ticket."

Biconditional Truth Table

- $p \leftrightarrow q$ means that p and q have the **same** truth value.
- Note this truth table is the exact **opposite** of $\oplus$'s!
  - $p \leftrightarrow q$ means $\neg(p \oplus q)$
- $p \leftrightarrow q$ does **not** imply p and q are true, or cause each other.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Boolean Operations Summary

- We have seen 1 unary operator and 5 binary operators. Their truth tables are below.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F
T	T	F	T	T	F	T	T

Well-formed Formula (WFF)

A **well-formed formula (Syntax of compound proposition)**

1. Any statement variable is a WFF.
2. For any WFF α, $\neg\alpha$ is a WFF.
3. If α and β are WFFs, then $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$ and $(\alpha \leftrightarrow \beta)$ are WFFs.
4. A finite string of symbols is a WFF only when it is constructed by steps 1, 2, and 3.

Example of well-formed formula

- By definition of WFF
  - WFF: $\neg(P \wedge Q)$, $(P \rightarrow (P \vee Q))$, $(\neg P \wedge Q)$,
    $((P \rightarrow Q) \wedge (Q \rightarrow R)) \leftrightarrow (P \rightarrow R))$, etc.
  - not WFF:
    1. $(P \rightarrow Q) \rightarrow (\wedge Q)$: $(\wedge Q)$ is not a WFF.
    2. $(P \rightarrow Q$: but $(P \rightarrow Q)$ is a WFF.

    etc..

Tautology

- **Definition**:

  A well-formed formula (WFF) is a ***tautology*** if for every truth value assignment to the variables appearing in the formula, the formula has the value of **true**.

  - *Ex.* $p \vee \neg p$ ($\models p \vee \neg p$)

Substitution instance

- **Definition**:

  A WFF A is a ***substitution instance*** of another formula B if A is formed from B by substituting formulas for variables in B under condition that the same formula is substituted for the same variable each time that variable is occurred.

  - *Ex*. B: $p \rightarrow (j \wedge p)$, A: $(r \rightarrow s) \rightarrow (j \wedge (r \rightarrow s))$

- **Theorem**:

  A substitution instance of a tautology is a tautology

  - *Ex*. B: $p \vee \neg p$, A: $(q \wedge r) \vee \neg (q \wedge r)$

Contradiction

- **Definition**:

 A WFF is a ***contradiction*** if for every truth value assignment to the variables in the formula, the formula has the value of **false.**

 Ex. $(p \land \neg p)$

Valid consequence (1)

- **Definition**:

  A formula (WFF) B is a ***valid consequence*** of a formula A, denoted by $A \models B$, if for all truth value assignments to variables appearing in A and B, the formula B has the value of true whenever the formula A has the value of true.

Valid consequence (2)

- **Definition**:

  A formula (WFF) B is a ***valid consequence*** of a formula $A_1,\ldots, A_n$($A_1,\ldots, A_n \models B$) if for all truth value assignments to the variables appearing in $A_1,\ldots, A_n$ and B, the formula B has the value of true whenever the formula $A_1,\ldots, A_n$ have the value of true.

Valid consequence (3)

- **Theorem**:

  $A \models B$ iff $\models (A \rightarrow B)$
- **Theorem**:

  $A_1, \ldots, A_n \models B$ iff $(A_1 \wedge \ldots \wedge A_n) \models B$
- **Theorem**:

  $A_1, \ldots, A_n \models B$ iff $(A_1 \wedge \ldots \wedge A_{n-1}) \models (A_n \rightarrow B)$

Logical Equivalence

- **Definition:**

  Two WFFs, p and q, are logically equivalent **IFF** p and q have the same truth values for every truth value assignment to all variables contained in p and q.

  $$Ex.\ \neg\neg p, p : \neg\neg p \Leftrightarrow p$$

  $$p \vee p, p \ : p \vee p \Leftrightarrow p$$

  $$(p \wedge \neg p) \vee q, q : (p \wedge \neg p) \vee q \Leftrightarrow q$$

  $$p \vee \neg p, q \vee \neg q : p \vee \neg p \Leftrightarrow q \vee \neg q$$

Logical Equivalence

- **Theorem**:

  If a formula A is equivalent to a formula B then $\models A \leftrightarrow B$ ($A \Leftrightarrow B$)
- **Theorem**:

  If a formula D is obtained from a formula A by replacing a part of A, say C, which is itself a formula, by another formula B such that $C \Leftrightarrow B$, then $A \Leftrightarrow D$

Proving Equivalence via Truth Tables

Ex. Prove that $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

Equivalence Laws - Examples

- *Identity*: $p \wedge \mathbf{T} \Leftrightarrow p \quad p \vee \mathbf{F} \Leftrightarrow p$
- *Domination*: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T} \quad p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
- *Idempotent*: $p \vee p \Leftrightarrow p \quad p \wedge p \Leftrightarrow p$
- *Double negation:* $\neg\neg p \Leftrightarrow p$
- *Commutative:* $p \vee q \Leftrightarrow q \vee p \quad p \wedge q \Leftrightarrow q \wedge p$
- *Associative:* $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

More Equivalence Laws

- *Distributive*: $p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$
  $p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$

- *De Morgan's*:
  $\neg(p \land q) \Leftrightarrow \neg p \lor \neg q$
  $\neg(p \lor q) \Leftrightarrow \neg p \land \neg q$

- *Trivial tautology/contradiction*:
  $p \lor \neg p \Leftrightarrow \mathbf{T}$ $\quad$ $p \land \neg p \Leftrightarrow \mathbf{F}$

Defining Operators via Equivalences

Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p \oplus q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$
  $p \oplus q \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$
- Implies: $p \rightarrow q \Leftrightarrow \neg p \vee q$
- Biconditional: $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
  $p \leftrightarrow q \Leftrightarrow \neg(p \oplus q)$

Exercise 1

- Let p and q be the proposition variables denoting

  p: It is below freezing.

  q: It is snowing.

  Write the following propositions using variables, p and q, and logical connectives.

a) It is below freezing and snowing. $p \wedge q$

b) It is below freezing but not snowing. $p \wedge \neg q$

c) It is not below freezing and it is not snowing. $\neg p \wedge \neg q$

d) It is either snowing or below freezing (or both). $p \vee q$

e) If it is below freezing, it is also snowing. $p \rightarrow q$

f) It is either below freezing or it is snowing, but it is not snowing if it is below freezing. $(p \vee q) \wedge (p \rightarrow \neg q)$

g) That it is below freezing is necessary and sufficient for it to be snowing $p \leftrightarrow q$

Predicate Logic

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.
- Propositional logic (recall) treats simple *propositions* (sentences) as atomic entities.
- In contrast, *predicate* logic distinguishes the *subject* of a sentence from its *predicate.*
  - *Ex.* arithmetic predicates: $x=3$, $x>y$, $x+y=z$
  - propositions: $4=3$, $3>4$, $3+4=7$

    if ($\underline{x>3}$) then $y=x$;

Universes of Discourse (U.D.s)

- The power of distinguishing objects from predicates is that it lets you state things about *many* objects at once.
- E.g., let $P(x)$="$x+1>x$". We can then say, "For *any* number x, $P(x)$ is true" instead of $(\mathbf{0}+1>\mathbf{0}) \wedge (\mathbf{1}+1>\mathbf{1}) \wedge (\mathbf{2}+1>\mathbf{2}) \wedge \ldots$
- *Definition:*
  - The collection of values that a variable x can take is called x's *universe of discourse*.

Quantifier Expressions

- *Definition:*
  - *Quantifiers* provide a notation that allows us to *quantify* (count) *how many* objects in the univ. of disc. satisfy a given predicate.
  - "$\forall$" is the FOR $\forall$LL or *universal* quantifier. $\forall x\, P(x)$ means *for all* x in the u.d., P holds.
  - "$\exists$" is the $\exists$XISTS or *existential* quantifier. $\exists x\, P(x)$ means there *exists* an x in the u.d. (that is, 1 or more) such that $P(x)$ is true.

The Universal Quantifier $\forall$

- Example:
  Let the u.d. of x be parking spaces at SNU.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *universal quantification of* $P(x)$, $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at SNU are full."
  - *i.e.*, "Every parking space at SNU is full."
  - *i.e.*, "For each parking space at SNU, that space is full."

The Existential Quantifier ∃

- Example:
  Let the u.d. of x be parking spaces at SNU.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *existential quantification of* $P(x)$, $\exists x\, P(x)$, is the *proposition*:
  - "Some parking space at SNU is full."
  - "There is a parking space at SNU that is full."
  - "At least one parking space at SNU is full."

Free and Bound Variables

- *Definition:*
  - An expression like $P(x)$ is said to have a *free variable* x (meaning, x is undefined).
  - A quantifier (either $\forall$ or $\exists$) *operates* on an expression having one or more free variables, and *binds* one or more of those variables, to produce an expression having one or more *bound variables*.
  - *Ex.* $\exists x\ [x+y=z]$, x is bound but y and z are free variables

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\ P(x,y)$ has 1 free variable y, and one bound variable x.
- "$P(x)$, where x=3" is another way to bind x.
  - *Ex.* "$x+y=3$", T if x=1 and y=2, F if x=2 and y=6
- An expression with zero free variables is a bona-fide (actual) proposition.
- An expression with one or more free variables is still only a predicate: $\forall x\ P(x,y)$

Nesting of Quantifiers

Example: Let the u.d. of x & y be people.

Let $L(x,y)$="x likes y" (a predicate w. 2 f.v.'s)

Then $\exists y\ L(x,y)$ = "There is someone whom x likes."
(A predicate w. 1 free variable, x)

Then $\forall x\ \exists y\ L(x,y)$ =
"Everyone has someone whom they like."
(A __Proposition__ with __0__ free variables.)

Example of Binding

- $\forall x \in \mathrm{I}\ [x<x+1]$: T
- $\forall x \in \mathrm{I}\ [x=3]$: F
- $\forall x \in \mathrm{I}\ \forall y \in \mathrm{I}\ [x+y>x]$: F
- $\forall x \in \mathrm{I}^+\ \forall y \in \mathrm{I}^+\ [x+y>x]$: T

- $\exists x \in \mathrm{I}\ [x<x+1]$: T
- $\exists x \in \mathrm{I}\ [x=3]$: T
- $\exists x \in \mathrm{I}\ [x=x+1]$: F

WFF for Predicate Calculus

A **WFF** for (the first-order) calculus

1. Every predicate formula is a WFF.
2. If P is a WFF, $\neg P$ is a WFF.
3. Two WFFs parenthesized and connected by $\wedge$, $\vee$, $\leftrightarrow$, $\rightarrow$ form a WFF.
4. If P is a WFF and x is a variable then $(\forall x)P$ and $(\exists x)P$ *are WFFs.*
5. A finite string of symbols is a WFF only when it is constructed by steps 1-4.

Quantifier Exercise

If $R(x,y)$="x relies upon y," express the following in unambiguous English:

$\forall x\, \exists y\, R(x,y)$ = Everyone has *someone* to rely on.

$\exists y\, \forall x\, R(x,y)$ = There's a poor overburdened soul whom *everyone* relies upon (including himself)!

$\exists x\, \forall y\, R(x,y)$ = There's some needy person who relies upon *everybody* (including himself).

$\forall y\, \exists x\, R(x,y)$ = Everyone has *someone* who relies upon them.

$\forall x\, \forall y\, R(x,y)$ = *Everyone* relies upon *everybody.* (including themselves)!

Natural language is ambiguous!

- "Everybody likes somebody."
  - For everybody, there is somebody they like,
    - $\forall x \, \exists y \, Likes(x,y)$ [Probably more likely.]
  - or, there is somebody (a popular person) whom everyone likes?
    - $\exists y \, \forall x \, Likes(x,y)$
- "Somebody likes everybody."
  - Same problem: Depends on context, emphasis.

More to Know About Binding

- $\forall x\ \exists x\ P(x)$ - x is not a free variable in $\exists x\ P(x)$, therefore the $\forall x$ binding isn't used.
- $(\forall x\ P(x)) \wedge Q(x)$ - The variable x in $Q(x)$ is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a proposition!

  $\forall x\ P(x) \wedge Q(x) \neq \forall x\ (P(x) \wedge Q(x))$

  $(\forall x\ P(x)) \wedge Q(x)$

  $\forall x\ P(x) \wedge Q(y)$: clearer notation
- $(\forall x\ P(x)) \wedge (\exists x\ Q(x))$ – This is legal, because there are 2 different x's!

Quantifier Equivalence Laws

- Definitions of quantifiers: If u.d.=a,b,c,…
  $\forall x\, P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\, P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$
- From those, we can prove the laws:
  $\neg\forall x\, P(x) \Leftrightarrow \exists x\, \neg P(x)$
  $\neg\exists x\, P(x) \Leftrightarrow \forall x\, \neg P(x)$
- Which *propositional* equivalence laws can be used to prove this? Demorgan's

Ex. $\neg\exists x\forall y\forall z\, P(x,y,z) \Leftrightarrow \forall x\neg\forall y\forall z\, P(x,y,z)$
$\Leftrightarrow \forall x\exists y\neg\forall z\, P(x,y,z)$
$\Leftrightarrow \forall x\exists y\exists z\, \neg P(x,y,z)$

More Equivalence Laws

- $\forall x\ \forall y\ P(x,y) \Leftrightarrow \forall y\ \forall x\ P(x,y)$
  $\exists x\ \exists y\ P(x,y) \Leftrightarrow \exists y\ \exists x\ P(x,y)$
- $\forall x\ \exists y\ P(x,y)$ <≠> $\exists y\ \forall x\ P(x,y)$

- $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\ P(x)) \wedge (\forall x\ Q(x))$
  $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow (\exists x\ P(x)) \vee (\exists x\ Q(x))$
- $\forall x\ (P(x) \vee Q(x))$ <≠> $(\forall x\ P(x)) \wedge (\forall x\ Q(x))$
  $\exists x\ (P(x) \wedge Q(x))$ <≠> $(\exists x\ P(x)) \vee (\exists x\ Q(x))$

Defining New Quantifiers

- Definition:
  - $\exists! x\ P(x)$ to mean "$P(x)$ is true of *exactly one* x in the universe of discourse."

- Note that $\exists! x\ P(x) \Leftrightarrow \exists x \big(P(x) \wedge \neg\exists y\ (P(y) \wedge (y \neq x))\big)$
  "There is an x such that $P(x)$, where there is no y such that P(y) and y is other than x."

Exercise 2

Let *F(x, y)* be the statement "*x loves y,*" where the universe of discourse for both *x* and *y* consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Jerry. $(\forall x)\ F(x, Jerry)$

b) Everybody loves somebody. $(\forall x)(\exists y)\ F(x,y)$

c) There is somebody whom everybody loves. $(\exists y)\ (\forall x)\ F(x,y)$

d) Nobody loves everybody. $\neg(\exists x)(\forall y)\ F(x,y)$

e) There is somebody whom Lydia does not love. $(\exists x)\ \neg F(Lydia,x)$

f) There is somebody whom no one loves. $(\exists x)(\forall y)\ \neg F(y,x)$

g) There is exactly one person whom everybody loves. $(\exists! x)(\forall y)\ F(y,x)$

h) There are exactly two people whom Lynn loves.

$$(\exists x)\ (\exists y)\ ((x \neq y) \wedge F(Lynn,x) \wedge F(Lynn,y) \wedge (\forall z)\ (F(Lynn,z) \rightarrow (z=x) \vee (z=y)))$$

i) Everyone loves himself or herself $(\forall x)\ F(x,x)$

j) There is someone who loves no one besides himself or herself.

$$(\exists x)\ (\forall y)\ (F(x,y) \leftrightarrow x=y)$$

Exercise

1. Let *p, q, and r be the propositions*

    p: You have the flu.

    q: You miss the final examination

    r: You pass the course

Express each of these propositions as an English sentence.

(a) $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$

(b) $(p \wedge q) \vee (\neg q \wedge r)$

Exercise (cont.)

2. Assume the domain of all people.

Let *J(x) stand for "x is a junior",*

S(x) stand for "x is a senior", and

L(x, y) stand for "x likes y".

Translate the following into well-formed formulas:

(a) All people like some juniors.

(b) Some people like all juniors.

(c) Only seniors like juniors.

Exercise (cont.)

3. Let *B(x) stand for "x is a boy", G(x) stand for "x is a girl", and T(x,y) stand for "x is taller than y".*

   Complete the well-formed formula representing the given statement by filling out the missing part.

(a) Only girls are taller than boys: $(?)(\forall y)((?\wedge T(x,y))\rightarrow ?)$

(b) Some girls are taller than boys: $(\exists x)(?)(G(x)\wedge(?\rightarrow ?))$

(c) Girls are taller than boys only: $(?)(\forall y)((G(x)\wedge ?)\rightarrow ?)$

(d) Some girls are not taller than any boy: $(\exists x)(?)(G(x)\wedge(?\rightarrow ?))$

(e) No girl is taller than any boy: $(?)(\forall y)((B(y)\wedge ?)\rightarrow ?)$